Exercise 5.1. Prove that for all $n$ every ideal of the ring $\mathbb{Z}/(n)$ is principal. Is $\mathbb{Z}/(n)$ a principal ideal domain? How would you determine the number of ideals of $\mathbb{Z}/(n)$?

Exercise 5.2. Let $R$ be an integral domain and let $r \in R$. Prove that $r$ is a unit if and only if $(r) = R$.

Exercise 5.3. Let $R$ be an integral domain. Prove that $0 \in R$ is not an irreducible. Also prove that if $u \in R$ is a unit and $r \in R$ is irreducible, then $au$ is irreducible.

Exercise 5.4. Let $n \in \mathbb{Z}$ such that $n \neq m^2$ for all $m \in \mathbb{Z}$. Prove that the function $N : \mathbb{Z}[\sqrt{n}] \to \mathbb{N}$ defined by

$$N(a + b\sqrt{n}) = a^2 - nb^2$$

satisfies

$$N(a + b\sqrt{n})N(c + d\sqrt{n}) = N((a + b\sqrt{n})(c + d\sqrt{n})).$$

Exercise 5.5. Let $R$ be an integral domain and $R' = R \setminus \{0\}$. Is it true or false that:

(a) if $|$ is an equivalence relation on $R'$, then $R$ is a field;

(b) if $x \sim y$ for all $x, y \in R'$, then $R$ is a field;

(c) if $a \sim b$ and $c \sim d$, then $ac \sim bd$;

(d) if $a \sim b$ and $c \sim d$, then $(a + c) \sim (b + d)$;

(e) if every element of $R'$ is a unit or a prime, then $R$ is a field?

Give a proof if the statement is true and a counter-example if it is false.

Exercise 5.6. Determine whether 17 is irreducible in each of $\mathbb{Z}, \mathbb{Z}[x], \mathbb{Z}[i], \mathbb{Z}[\sqrt{10}]$.

Exercise 5.7. Find two different factorisations of 10 in $\mathbb{Z}[\sqrt{-6}]$. Find an irreducible element of $\mathbb{Z}[\sqrt{-6}]$ that is not prime.
Exercise 5.8. Show that 8 can be written as both the product of 2 irreducibles and as the product of 3 irreducibles in \( \mathbb{Z}[\sqrt{-7}] \).

Exercise 5.9. Let \( R \) be a ring and \( I \) an ideal of \( R \) such that \( R/I \) is an integral domain. Prove that \( I \) is a prime ideal (that is, prove the direct implication of Theorem 9.2).

Exercise 5.10. Let \( R \) be the ring with elements \( \{ a/b \in \mathbb{Q} : b \equiv 1 \text{ or } 2 \pmod{3} \} \) and the usual operations \( + \) and \( \ast \) in \( \mathbb{Q} \) and let \( I = \{ a/b \in R : a \equiv 0 \pmod{3} \} \). Then

(i) prove that \( R \) is a subring of \( \mathbb{Q} \);
(ii) prove that \( I \) is an ideal in \( R \);
(iii) prove that \( R/I \) is a field.

Exercise 5.11. Prove that \( \mathbb{Z}[\sqrt{2}] \) and \( \mathbb{Z}[\sqrt{3}] \) are not isomorphic rings.

Exercise 5.12. Let \( \alpha \in \mathbb{Z}[i] \) be prime. Prove that precisely one of the following holds:

(i) \( \alpha \sim 1 + i \);
(ii) \( \alpha \sim p \) where \( p \in \mathbb{Z} \) is prime and \( p \equiv 3 \pmod{4} \);
(iii) \( \alpha \) divides a prime \( p \in \mathbb{Z} \) with \( p \equiv 1 \pmod{4} \).

Exercise 5.13. Let \( R \) be a principal ideal domain and let \( I \) be an ideal of \( R \). Prove that \( R/I \) is a principal ideal domain.

Find an example of a ring that is not an integral domain but where every ideal is principal.

Exercise 5.14. Let \( I \) and \( J \) be ideals and \( K \) be a prime ideal of a ring \( R \). Prove that if \( IJ \subseteq K \), then \( I \subseteq K \) or \( J \subseteq K \).

Exercise 5.15. Let \( R \) and \( S \) be rings and let \( f : R \rightarrow S \) be a homomorphism. Prove that \( \ker(f) \) is a prime ideal if \( S \) is an integral domain. Prove that \( \ker(f) \) is a maximal ideal if \( S \) is a field and \( f \) is surjective.