Exercise 3.1. Determine which of the following are subrings of the given rings.

(i) the positive integers in \( \mathbb{Z} \);

(ii) all polynomials with integer constant in \( \mathbb{Q}[x] \);

(iii) all integers divisible by 3 in \( \mathbb{Z} \);

(iv) all polynomials of degree at least 6 in \( \mathbb{Q}[x] \);

(v) the set \( \{ 75a + 30b : a, b \in \mathbb{Z} \} \) in \( \mathbb{Z} \);

(vi) all the zero divisors of \( \mathbb{Z}/(14) \) in \( \mathbb{Z}/(16) \).

Also determine which of the examples above are ideals in the respective rings.

Exercise 3.2. Let \( R \) denote the set of all subsets of a set \( S \). Define operations \(+\) and \(*\) on \( R \) by

\[ A + B = (A \cup B) \setminus (A \cap B) \text{ and } A * B = A \cap B, \]

where \( A, B \in R \). Prove that \( R \) is a ring. [Aside: \( R \) is called a Boolean ring.]

Does this ring have an identity element? Which elements of the ring have multiplicative inverses? If we redefine \(+\) by \( A + B = A \cup B \), do we still get a ring?

Let \( A \) be a subset of \( S \). Describe the ideal of \( R \) generated by \( A \).

Exercise 3.3. Prove that the set of real polynomials \( a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \) where \( a_0 = a_1 = 0 \) is a subring of the polynomial ring \( \mathbb{R}[x] \). Is it an ideal?

Exercise 3.4. Prove that the set of all real polynomials \( a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \) for which the sum \( a_0 + a_1 + a_2 + \cdots + a_n = 0 \) is an ideal of \( \mathbb{R}[x] \).

Exercise 3.5. Prove that the set \( \{ r + s \sqrt{2} : r, s \in \mathbb{Q} \} \) is a field under real addition and multiplication. Prove that it is the smallest subfield of \( \mathbb{R} \) which contains \( \sqrt{2} \).

Exercise 3.6. What is the ideal of \( \mathbb{R} \) generated by \( \sqrt{2} \)?

Exercise 3.7. If \( R \) is a commutative ring with identity whose only ideals are \( \{0\} \) and \( R \), prove that \( R \) is a field. If \( R \) is a commutative ring with identity, do the non-invertible elements of \( R \) form an ideal? Prove this or find a counterexample.
Exercise 3.8. Let \( R \) be the set of real matrices of the form
\[
\begin{pmatrix}
a & b \\
2b & a
\end{pmatrix}.
\]
Prove that \( R \) is a subring of the ring of all real matrices. If we insist that the entries of \( R \) are rationals, prove that \( R \) is then a field. [Hint: a matrix with entries in a field is invertible if its determinant is non-zero.]

If the entries of \( R \) are taken from the ring \( \mathbb{Z}/(3) \), prove that \( R \) is a field with 9 elements.

Exercise 3.9. Prove Lemma 5.13 from lectures.

Exercise 3.10. Prove that every field is a PID.

Exercise 3.11. Let \( I \) and \( J \) be ideals in a commutative ring \( R \) with identity. Prove that \( I \cap J \), \( I + J = \{ i + j : i \in I, j \in J \} \), and
\[
IJ = \left\{ \sum_{i=1}^{n} a_ib_i : n \geq 1, a_i \in I, b_i \in J \right\}
\]
are ideals in \( R \).

Prove that \( IJ \subseteq I \cap J \). Find examples of ideals \( I \) and \( J \) such that \( IJ \neq I \cap J \). Is \( \{ ij : i \in I, j \in J \} \) an ideal?

Exercise 3.12. Let
\[
I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots
\]
be an infinite increasing sequence of ideals in a ring \( R \). Prove that the union of the ideals is an ideal. Show that the union
\[
\{ 2m : m \in \mathbb{Z} \} \cup \{ 3n : n \in \mathbb{Z} \}
\]
of two ideals in \( \mathbb{Z} \) is not even a subring of \( \mathbb{Z} \).

Exercise 3.13. Let \( R \) be a ring with the property that every ideal \( I \subseteq R \) is finitely generated, that is, there exist \( r_1, \ldots, r_n \in R \) where \( I = (r_1, r_2, \ldots, r_n) \). A ring with this property is called noetherian. Let
\[
I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots
\]
be an infinite increasing sequence of ideals in a ring \( R \). Prove that there exists \( N \in \mathbb{N} \) such that \( I_N = I_{N+1} = \cdots \).

Exercise 3.14. Let \( I \) be an ideal in a ring \( R \). Prove that \( I[x] \) is an ideal in \( R[x] \).