Exercise 2.1. Let $R$ be any ring. Use the Ring Axioms to prove that $0 \ast a = 0 = a \ast 0$ for any $a \in R$.

If $R$ has a multiplicative identity $1$, prove that $(-1) \ast (-1) = 1$.

Exercise 2.2. Let $\mathbb{R}$ denote the real numbers and $R$ be the set of mappings $f : \mathbb{R} \to \mathbb{R}$. Define $+$ and $\ast$ on $R$ by

$$
(f + g)(x) = f(x) + g(x)
$$

$$
(f \ast g)(x) = f(x) \ast g(x)
$$

for all $x \in \mathbb{R}$. Prove that $R$ is a commutative ring with identity. Which elements of $R$ have multiplicative inverses? Does $R$ have zero divisors? Is $R$ a ring when multiplication $\ast$ is defined to be composition of mappings?

Exercise 2.3. Let $R$ be a set with operations $+$ and $\ast$ satisfying A2 and M2. Prove that if $1 = 0$ in $R$, then $R$ is a ring with one element.

Exercise 2.4. Let $R$ be a ring satisfying M2 and M3. Prove that $R$ has no zero divisors.

Exercise 2.5. Let $R = \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$. Define $+$ and $\ast$ by

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac, bd).$$

Prove that $R$ is a commutative ring with identity. Find the elements of $R$ that have a multiplicative inverse.

Exercise 2.6. Prove that the set $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is a ring under real addition and multiplication. Find the multiplicative inverses of the elements $1 + \sqrt{2}$ and $3 + 2\sqrt{2}$.

Show that $a + b\sqrt{2} \in R$ is a unit if and only if $a^2 - 2b^2 = 1$. Deduce that $R$ is not a division ring and hence not a field.

Exercise 2.7. Let $M$ be the set of matrices

$$
\left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.
$$

Show that $M$ is a ring under the operations of matrix addition and multiplication. Is $M$ a commutative ring? Does $M$ have an identity?
Exercise 2.8. Let $\mathbb{H}$ denote the quaternions as defined in Example 2.6. From the multiplication table defined in Example 2.6, we deduce that

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.1}$$

Prove that it is possible to deduce the multiplication table from the equalities in (2.1).

Exercise 2.9. Let $x \in \mathbb{Z}/(n)$ with $x \neq 0$. Prove that $x$ is a unit if and only if $x$ is coprime to $n$. Deduce that $\mathbb{Z}/(n)$ is a division ring if and only if $n$ is a prime.

Exercise 2.10. Let $a + bi + cj + dk$ be a non-zero element in the quaternions $\mathbb{H}$. Prove that

$$\frac{1}{a^2 + b^2 + c^2 + d^2}$$

is the multiplicative inverse of $a + bi + cj + dk$. Deduce that $\mathbb{H}$ is a division ring.

Prove that $\mathbb{H}$ is not an integral domain.

Exercise 2.11. Prove the following statements:

(i) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ are division rings;

(ii) $\mathbb{Z}$ is not a division ring;

(iii) the ring $M_2(\mathbb{R})$ is not a division ring.

Exercise 2.12. Let $R$ be a ring and $a, b \in R \setminus \{0\}$ with $ab = 0$. Prove that neither $a$ nor $b$ is a unit.

Exercise 2.13. Prove that the set $C[0, 1]$ of real-valued continuous functions on the interval $[0, 1]$ is a ring under the operations $+$ and $\ast$. What are the one and zero in $C[0, 1]$? Is $C[0, 1]$ commutative, or an integral domain? Does $C[0, 1]$ have zero divisors? What are the units in $C[0, 1]$? Is $C[0, 1]$ a division ring or a field?

Exercise 2.14. Let $R$ be a ring and let $M_n(R)$ denote the set of $n \times n$ matrices with entries in $R$. Prove that $M_n(R)$ is a ring.

Exercise 2.15. Let $R$ be a set with operations $+$ and $\ast$ satisfying the Ring Axioms except A4 and including M2. Prove that A4 holds. [Hint: Expand $(1 + 1) \ast (a + b)$ using A2, A3 and D in two different ways.]

Exercise 2.16. Let $R$ be a ring where $x^2 = x \ast x = x$ for all $x \in R$. Prove that $R$ is a commutative ring where $2x = x + x = 0$ for all $x$.

Exercise 2.17. Is it true that $(x + y)^2 = x^2 + 2x \ast y + y^2$ in all rings?