LMS Short Course on Computational Group Theory
Lab session 3
Using group libraries and conducting searches

1. How many non-abelian groups of order 24 are there? (Of course, with this formulation we always mean “up to isomorphism”.)
   **Hint**: You want to use the small groups library, \(\rightarrow\) ?NumberSmallGroups, \(\rightarrow\) ?AllSmallGroups, \(\rightarrow\) ?IsAbelian

2. How many non-abelian groups of order 128 are there?
   Compute the average of the sizes of their centres.
   **Hint 1**: To automate the counting you can use \(\rightarrow\) ?List and \(\rightarrow\) ?Collected or use \(\rightarrow\) ?Filtered to get them all
   **Hint 2**: For the centres the \(\rightarrow\) ?arrow notation could be convenient

3. What ID does the group generated by the three permutations
   \((2, 4, 6, 8, 10), (1, 9)(2, 8)(3, 7)(4, 6)\) and \((1, 6)(2, 7)(3, 8)(4, 9)(5, 10)\)
   have?
   **Hint**: \(\rightarrow\) ?IdGroup, \(\rightarrow\) ?SmallGroup

4. Find all composition series of all non-solvable groups of order 120.
   **Hint 1**: You can avoid making all small groups of order 120 and then Filtered by using the more sophisticated syntax of \(\rightarrow\) ?AllSmallGroups
   **Hint 2**: Simply compute all normal subgroups of them using \(\rightarrow\) ?NormalSubgroups

5. How many elements of order 3 do all groups of order 48 have together? (Of course, we mean to take one group of each isomorphism type.)
   **Hint 1**: Fetch them all using ?AllSmallGroups, for each of them, ask for all elements (\(\rightarrow\) ?Elements) and let GAP count.
   **Hint 2**: If you have a list \(L\) of groups, you can use a \(\rightarrow\) ?for loop to run through all of them like this:
   ```gap
   for g in L do
     do stuff with g
   od;
   **Hint 3**: The same technique can be used to run through a list of elements. Use \(\rightarrow\) ?if to decide, whether or not an element has order 3 as in
   ```gap
   if Order(x) = 3 then
     increase a counter
   fi;
   ```

6. Show that the Sylow 2-subgroups of the Mathieu group \(M_{24}\) and the sporadic simple Held group \(He\) are isomorphic.
   **Hint 1**: \(\rightarrow\) ?MathieuGroup
   **Hint 2**: Use the AtlasRep package and \(\rightarrow\) ?AtlasGroup to fetch generators of \(He\) from the internet.
   **Hint 3**: Use \(\rightarrow\) ?SmallerDegreePermutationRepresentation for the Sylow 2-subgroup of \(He\)
   **Hint 4**: Use \(\rightarrow\) ?IsomorphismGroups

7. Find the group with the fewest elements that is non-abelian, has trivial center and contains an element \(a\) of order 2 and an element \(b\) of order 3 such that \(ab\) has order 5.