Recognising Matrix Groups

Max Neunhöffer

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Tucson 2006
All of this is joint work with Ákos Seress.
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Lots of others contributed ideas, results, and code.
The Problem

$\mathbb{F}_q$ field with $q$ elements
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$\{M_1, M_2, \ldots, M_k\} \subseteq \text{GL}_d(\mathbb{F}_q)$
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$G := \langle M_1, M_2, \ldots, M_k \rangle$ finite
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Questions

- What is $|G|$?
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Constructive recognition
Straight line programs
Efficiency
Discrete logarithm problem
History

Some Solutions

What one can do
The composition tree
An example: Low index
Aschbacher classes
Leaves

State of implementation
GAP packages recog and recogmethods
Help is appreciated

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- Given $g \in G$, write $g$ as product of the $M_i$
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- What is \(|G|\)?
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  (or in terms of some “nice” generating set of \(G\)).
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We call this “constructive recognition of $G$”. 
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Variant: $\{\bar{M}_1, \ldots, \bar{M}_k\} \subseteq \text{PGL}_d(\mathbb{F}_q)$, $G := \langle \bar{M}_1, \ldots, \bar{M}_k \rangle$
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Straight line programs

Example:

```
# input:
r := [ a, b, c ];
# program:
# return values:
[ r[4], r[5]^5 ]
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Straight line programs (SLPs) only reference earlier results, do not contain loops, branches or subroutines, and can express long products memory efficiently.
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- do not contain loops, branches or subroutines, and
- can express long products memory efficiently.
Efficiency

What does “efficiently” mean?

The maximal number of operations necessary is bounded by a (fixed) polynomial in the "input size". The input size is measured by $d$: size of matrices, $k$: number of matrices, and $\log(q)$: size of a field element. This is called "in polynomial time". Also the length of the resulting straight line programs should be decent.
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\[\implies\] this decision shortened SLPs from 500,000 steps down to 500 in examples
Nasty special case

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$\implies$ We work “modulo” this problem.
History

The **Matrix Group Recognition Project**:

- 1988, Oberwolfach, Joachim Neubüser:
  How to decide, whether $G = \text{GL}_d(q)$?

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- A new implementation in GAP
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- A new implementation in GAP
- Go for completely analysed polynomial-time algorithms
- Improve algorithms
What one can do with matrices

With a matrix group $G = \langle M_1, \ldots, M_k \rangle \leq \text{GL}_d(q)$ we can

- multiply, invert, compare, power up matrices
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- act with matrices on vectors or on subspaces
  $\rightarrow$ gives homomorphisms to permutation groups
Homomorphisms

**Try reduction:** For \( G = \langle M_1, \ldots, M_k \rangle \leq \text{GL}_d(q) \) find a homomorphism \( \varphi : G \rightarrow H \) which is

- explicitly computable
Homomorphisms

Try reduction: For $G = \langle M_1, \ldots, M_k \rangle \leq \text{GL}_d(q)$ find a homomorphism $\varphi : G \rightarrow H$ which is

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Assume we can constructively recognise $H$.

Set $N := \ker(\varphi)$. Then:
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- map $g$ to $H$ via $\varphi$
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- create a (pseudo-) random element $g$ in $G$
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- execute $S$ on $M_1, \ldots, M_k$, get $g' \in G$ s.t. $\varphi(g) = \varphi(g')$
- $\Rightarrow g^{-1} \cdot g' \in N$
- this creates a (pseudo-) random element in $N$
Composition trees

Produce generators of $N := \ker(\varphi)$ and recognise.
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- We have recognised $G$ constructively!
Get the recursion going ... 

Choose as “nice generators” $M'_1, \ldots, M'_k$ for $G$:

- preimages under $\varphi$ of the nice generators of $H$ plus
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- express $g'^{-1} \cdot g \in N$ as SLP $S''$ in $N$
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- express $g'^{-1} \cdot g \in N$ as SLP $S''$ in $N$
- put together $S$ from $S'$ and $S''$ plus one multiplication
A Composition Tree

G

N H

N H N H

N H

N H

1 1 2 2

3 3

G

N H

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N H

N H

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3 3

N1 H1 N2 H2

N3 H3

Upward arrows: monomorphisms
Downward arrows: epimorphisms
Low index

Assume:

- $G$ has a maximal subgroup $K$ of low index
- $G$ acts irreducibly
- $K$ leaves a subspace $0 < W < \mathbb{F}_q^{1 \times d}$ invariant
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- create random elements, hope they generate $K$
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- calculate its orbit under the action of $G$
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- $G$ acts irreducibly
- $K$ leaves a subspace $0 < W < \mathbb{F}_q^{1 \times d}$ invariant

Try to find a homomorphism in the following way:
- **create** random elements, hope they generate $K$
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Unfortunately, it is not yet analysed to be polynomial-time!
Aschbacher’s Theorem

Aschbacher classified the maximal subgroups of $GL_d(q)$. 

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The Problem

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Straight line programs
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Some Solutions

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The composition tree
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**Theorem (Aschbacher, 1984)**

If $G < \text{GL}_d(q)$ then it falls under at least one of:

- **C1** $G$ leaves invariant a subspace $0 < W < \mathbb{F}_q^{1 \times d}$
- **C2** $G$ preserves a decomposition $\mathbb{F}_q^{1 \times d} \cong V_1 \oplus \cdots \oplus V_j$
- **C3** $G$ comes from a bigger field (semilinear)
- **C4** $G$ preserves a decomposition $\mathbb{F}_q^{1 \times d} \cong V_1 \otimes V_2$
- **C5** $G$ is realizable over a subfield
- **C6** $G \leq N_{\text{GL}}(r^{1+2k})$ where $r^{1+2k}$ is an extraspecial group
- **C7** $G$ is tensor-induced
- **C8** $G$ contains a “classical group” like $\text{SL}_d(q)$ or $\text{Sp}_d(q)$
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All classes C1 to C7 are defined “geometrically” and promise some kind of homomorphism or “simplification”.

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Problem children

The leaves are a problem: Need representation theory.
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Classify: Irred. modular representations of finite groups.
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- recognise the group for example by looking at distribution of element orders of random elements (“non-constructive recognition”)
- use collected data about representations or
- use collected data about subgroups
- directly recognise the group constructively:
  - use base and strong generating sets (matrix Schreier-Sims)
  - use tricks involving involution centralisers
The GAP package recog

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- the infrastructure for SLPs, matrix handling, etc.
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Recognising Matrix Groups

Max Neunhöffer

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**Authors:** MN and Ákos Seress
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Authors: (currently)
Peter Brooksbank, Maska Law, Steve Linton, MN, Alice Niemeyer, Eamonn O’Brien, Ákos Seress.
Still missing

• analysis of the low index procedure
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- a whole lot of documentation
- higher level algorithms after recognition (Sylow subgroups, maximal subgroups, centralisers, normalisers, etc.)
Help is appreciated

Everybody is welcome to contribute.

We need ideas, code, and analysis.