1 The algebras

Let \( n \in \mathbb{N}, \, W := \text{sym group on } \{1, 2, \ldots, n\}, \, s_i := (i, i + 1), \, S := \{s_1, s_2, \ldots, s_{n-1}\} \):

\[
\Rightarrow W = \langle s_1, \ldots, s_{n-1} \rangle = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = \text{id}, s_is_j = s_js_i \text{ for } |i - j| \geq 2, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } 1 \leq i < n \rangle
\]

Call \( w = t_1 \cdots t_k \) with \( t_i \in S \) “reduced expression” if shortest possible.

**Def.:** Let a com. ring, \( v \in A \) invertible.

\( \mathcal{H}_n(A, v) := \text{assoc. } A\text{-alg. with generators } \{T_s \mid s \in S\} \text{ subject to relations:} \)

- \((T_s - v)(T_s + v^{-1}) = 0 \) for \( s \in S \)
- \( T_sT_t = T_tT_s \) for \( s, t \in S \) and \( st = ts \) and \( s \neq t \)
- \( T_sT_sT_s = T_sT_sT_s \) for \( s, t \in S \) and \( st \neq ts \).

This is called the “Iwahori-Hecke algebra of type \( A_{n-1} \) over \( A \) with parameter \( v \)”.

**Remark:** For \( v = 1: \mathcal{H}_n(A, 1) = AW \) the group algebra.

**Theorem (Bourbaki):** For \( w \in W \), the element \( T_w := T_{t_1} \cdots T_{t_2} \cdots T_{t_k} \) for any reduced expression \( w = t_1 \cdots t_k \) with \( t_i \in S \) is well-defined and \( \{T_w \mid w \in W\} \) is an \( A \)-basis of \( \mathcal{H}_n(A, v) \).

**Corollary:** If \( \varphi : A \to A' \) is a homomorphism of com. rings, then there is a ring homomorphism

\[
\varphi : \mathcal{H}_n(A, v) \to \mathcal{H}_n(A', \varphi(v)) \quad \text{such that } \forall w \in W
\]

This is called a “specialisation”, because:

- If \( A = \mathbb{Z}[x, x^{-1}] \), then \( \mathcal{H}_n(\mathbb{Z}[x, x^{-1}], x) \) is called the “generic Iwahori-Hecke Algebra of type \( A_{n-1} \)”.

Possible specialisations:

- \( \mathcal{H}_n(\mathbb{Z}[x, x^{-1}], x) \to \mathcal{H}(\mathbb{Q}(x), x) \) using \( \mathbb{Z}[x, x^{-1}] \subseteq \mathbb{Q}(x) \)
- \( \mathcal{H}_n(\mathbb{Z}[x, x^{-1}], x) \to \mathcal{H}(\mathbb{Q}(\zeta_n), \zeta_n) \) using \( \mathbb{Z}[x, x^{-1}] \to \mathbb{Q}(\zeta_n), x \mapsto \zeta_n \in \mathbb{Q}(\zeta_n) \)
- \( \mathcal{H}_n(\mathbb{Z}[x, x^{-1}], x) \to \mathcal{H}(\mathbb{F}_p, u) \) using \( \mathbb{Z}[x, x^{-1}] \to \mathbb{F}_p, x \mapsto u, z \mapsto z + p \mathbb{Z} \)

2 Representations and modules

Let \( A \) be a field, \( \mathcal{H} := \mathcal{H}_n(A, v) \).

**Def.:** \( M \) finite dim. \( A \)-vector space. A **representation** of \( \mathcal{H} \) is a homomorphism \( \varphi : \mathcal{H} \to \text{End}_A(M) \) of \( A \)-algebras where \( \text{End}_A(M) = \{\alpha : M \to M \mid \alpha \text{ is } A\text{-linear}\} \).

\((M, \varphi)\) is called an \( \mathcal{H} \)-**module**. Instead of \( \varphi(h)(m) \) for \( h \in \mathcal{H} \) and \( m \in M \) write \( hm = h \cdot m \), i.e.: \( h(m + m') = hm + hm' \) and \( (h \cdot h')(m) = h \cdot (h'\cdot m) \) for all \( h, h' \in \mathcal{H} \) and \( m, m' \in M \).

[“Representation theory” studies and classifies the modules of an algebra (up to isomorphism)]

**Ex.:** Let \( M_1 \) be a 1-dim. v.s. over \( A \), then \( T_v \mapsto v \) (note \( \text{End}_A(M_1) \cong A \)) is a representation.

Let \( M \) be a 1-dim. v.s. over \( A \), then \( T_v \mapsto -v^{-1} \) is another one.

**Def.:** \( M \) an \( \mathcal{H} \)-module. A **submodule** is a sub vector space \( N \leq M \), such that \( hn \in N \forall h \in \mathcal{H} \forall n \in N \).

[ \( \mathcal{H} \)-invariant space ]

\( M \) is called **irreducible**, if it has only the submodules \( 0 \) and \( M \).

If \( N \leq M \) is a submodule, then \( M/N \) is an \( \mathcal{H} \)-module.

A **composition series** of \( M \) is a chain

\[
0 = N_0 < N_1 < \cdots < N_r = M
\]

of submodules of \( M \), such that \( N_i/N_{i-1} \) is irreducible \( \forall 1 \leq i \leq r \).

[“Representation theory” begins by classifying the irreducible modules.]

**Def.:** A **partition** \( \lambda \) of \( n \) is a tuple \( (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \) and \( \sum_{i=1}^{k} \lambda_i = n \). Write \( \lambda \vdash n \).

**Theorem (Richard Dipper, Gordon James 1986)**

Define combinatorically for \( \lambda \vdash n \) a “Specht-module” \( S^\lambda_n \) for \( \mathcal{H} = \mathcal{H}_n(A, v) \), equipped with a bilinear form \( \beta^\lambda \) : \( S^\lambda_n \times S^\lambda_n \to A \) such that \( \beta^\lambda(hx, hy) = h \beta^\lambda(x, y) \) for all \( h \in \mathcal{H} \) and all \( x, y \in S^\lambda_n \).

Set \( \text{rad}(\beta^\lambda) := \{x \in S^\lambda_n \mid \beta^\lambda(x, y) = 0 \forall y \in S^\lambda_n \} \leq S^\lambda_n \). Set \( D^\lambda_n := S^\lambda_n / \text{rad}(\beta^\lambda) \).
Let $e := e(v)$ be the least natural number such that $1 + v + v^2 + \cdots + v^{e-1} = 0$ (or $e := \infty$ if none exists).

[This is the multiplicative order of $v$]

Then:
If $e > n$, then $\text{rad}(\beta_\lambda) = 0$ and $D_\lambda = S_\lambda$ is irreducible. [easy, semisimple case, dimensions known]
If $e \leq n$, then $D_\lambda$ is either irreducible or 0.

Up to isomorphism, $\{D_\lambda \mid \lambda \vdash n, D_\lambda \neq 0\}$ are all simple modules. [\(\lambda = (n)\) and $\lambda = (1^n)$]

### 3 The conjecture

[Everything solved?? NO!]

**Major open problem:** For $A = \mathbb{F}_p$, determine $\dim_A(D_\lambda)$ for $e \leq n$.

**Equivalent:** Determine $\left| S_\lambda : D_\mu \right| := \#$ of occurrences of $D_\mu$ as factor in a composition series $0 = N_0 < N_1 < \cdots < N_r = S_\lambda$, (i.e. $\{|i \mid N_i/N_{i-1} \cong D_\mu\}$, called “decomposition numbers”).

**Remark:** E.g. for $A = \mathbb{F}_p$, $p \leq n$, $v = 1 \implies e = p$ since $1^0 + 1^1 + 1^2 + \cdots + 1^{p-1} = 0$.

→ modular (i.e. char$A = p$) representations of the sym. group.

**Conjecture (James, 1990)** Consider $v \in \mathbb{F}_p$, $e := e(v)$ such that $e \cdot p > n$. Then for $\lambda, \nu \vdash n$:

$$\left| S_\lambda : D_\mu \right| = \left| S_\lambda : D_\mu \right|$$

for $\mathcal{H}_n(\mathbb{F}_p, v)$ for $\mathcal{H}_n(\mathbb{Q}(\zeta_e), \zeta_e)$

with $\zeta_e \in \sqrt{TC}$ primitive, i.e. decomposition numbers depend on $e(v)$ but not on $p$ and $e$!

[\(\implies\) proved in some cases]

E.g.: $\exists$ limit $N$ such that it holds for all $p > N$. **But no explicit limit known!**

### 4 An equivalent statement

**Def.:** A field, $(\mathbb{Z}, +)$ a totally ordered abelian group. Then $\nu : A \to \mathbb{Z} \cup \{\infty\}$ is called a valuation, if the following hold:

\[
\begin{align*}
\nu(a) &= \infty \iff a = 0 \\
\nu(a \cdot b) &= \nu(a) + \nu(b) \forall a, b \in A \\
\nu(a + b) &\geq \min\{\nu(a), \nu(b)\}
\end{align*}
\]

**Remark:** It follows, that $R := \{a \in A \mid \nu(a) \geq 0\}$ is a subring of $A$ with unique maximal ideal $J := \{a \in A \mid \nu(a) > 0\}$, since $\nu(1) = 0$ and thus $\nu(a^{-1}) = -\nu(a)$. $R$ is a valuation ring in $A$, which means: $\forall 0 \neq a \in A$ we have: $a$ or $a^{-1}$ or both lie in $R$.

**Ex.:** Let $A := \mathbb{Q}$, $p$ a prime. Define for $a \in \mathbb{Z} \setminus \{0\}$: $\nu(a) := k$ if $p^k \mid a$ but $p^{k+1} \nmid a$. For $a/b \in \mathbb{Q}$ set $\nu(a/b) := \nu(a) - \nu(b)$.

This gives a valuation on $\mathbb{Q}$ with values in $\mathbb{Z} \cup \{\infty\}$.

**Valuation ring** $R = \{a/b \mid \gcd(a, b) = 1 \text{ and } p
\}> b\}$

**max. ideal** $J = \{ap/b \mid \gcd(a, b) = 1 \text{ and } p \mid b\}$

**factor ring** $R/J \cong \mathbb{F}_p$

**Def.:** Call an idempotent $f^2 = f$ primitive iff it cannot be written $f = f_1 + f_2$ with two orthogonal idempotents $f_1^2 = f_1$ and $f_2^2 = f_2$ and $f_1f_2 = 0 = f_2f_1$.

Assume the situation of James’ conjecture: $v \in \mathbb{F}_p$, $e = e(v), e \cdot p > n$ and $\lambda, \mu \vdash n$.

I have defined two valuations

$\nu_1 : \mathbb{Q}(x) \to \mathbb{Z} \cup \{\infty\}$ and $\nu_2 : \mathbb{Q}(x) \to (\mathbb{Z} \times \mathbb{Z}) \cup \{\infty\}$

with valuation rings $\mathcal{R}_1 = \{f \in \mathbb{Q}(x) \mid \nu_1(f) \geq 0\}$ and maximal ideals $J_i := \{f \in \mathbb{Q}(x) \mid \nu_i(f) > 0\}$ such that $\mathcal{R}_1/J_1 \cong \mathbb{Q}(\zeta_e)$ and $\mathcal{R}_2/J_2 \cong \mathbb{F}_p$.

Then: $\mathcal{R}_2 \subset \mathcal{R}_1 \subset \mathbb{Q}(x)$ and thus $\mathcal{H}_n(\mathcal{R}_2, x)$ embeds into $\mathcal{H}_n(\mathcal{R}_1, x)$.

**Theorem (N, 2003)**

James’ conjecture holds if and only if every primitive idempotent of $\mathcal{H}_n(\mathcal{R}_2, x)$ is primitive as idempotent of $\mathcal{H}_n(\mathcal{R}_1, x)$. 

2