Recognising Matrix Groups

Max Neunhöffer

Lehrstuhl D für Mathematik
RWTH Aachen

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All of this is joint work with Ákos Seress.

Lots of others contributed ideas, results, and code.
The Problem

$\mathbb{F}_q$ field with $q$ elements

$\{M_1, M_2, \ldots, M_k\} \subseteq \text{GL}_d(\mathbb{F}_q)$

$G := \langle M_1, M_2, \ldots, M_k \rangle$ finite

Questions

- What is $|G|$?
- What can be said about the isomorphism type?
- Given $g \in G$, write $g$ as product of the $M_i$ (or in terms of some “nice” generating set of $G$).
- Do all this “efficiently”.

We call this “solving the word problem in $G$”.
Straight line programs

Example:

```plaintext
# input:
r := [ a, b, c ];
# program:
# return values:
[ r[4], r[5]^5 ]
```

Executed with input \((a, b, c)\) this returns:

\((a^2ba^{-2}, a^2ba^{-2}c^7 a^2ba^{-2}c^7 a^2ba^{-2}c^7 a^2ba^{-2}c^7)\)

Straight line programs (SLPs)

- only reference earlier results,
- do not contain loops, branches or subroutines, and
- can express long products memory efficiently.
Efficiency

What does “efficiently” mean?

The maximal number of operations necessary is bounded by a (fixed) polynomial in the “input size”.

The input size is measured by

- \(d\): size of matrices,
- \(k\): number of matrices, and
- \(\log(q)\): size of a field element.

This is called “in polynomial time”.

Also the length of the resulting straight line programs should be decent.

\[\implies \text{we use a “nice” generating set} \]

\[\implies \text{this decision shortened SLPs from 500,000 steps down to 500 in examples} \]
Nasty special case

Is there hope?

$q$ large, $d = k = 1$, $M_1 = [\zeta]$ with $\zeta$ a primitive root of $\mathbb{F}_q$

Then our task is the Discrete Logarithm Problem to which there is currently

NO SOLUTION KNOWN in polynomial time in $\log(q)$

$\implies$ We work “modulo” this problem.
History

The Matrix Group Recognition Project:

- 1988, Oberwolfach, Joachim Neubüser: How to decide, whether $G = \text{GL}_d(q)$?
- 1992, Peter Neumann, Cheryl Praeger: Algorithm to decide whether $\text{SL}_d(q) \leq G$.
- Eamonn O’Brien: Implementation in Magma
- Lots of other people . . .

Our Goals:

- A new implementation in GAP
- Go for completely analysed polynomial-time algorithms
- Improve algorithms
What one can do with matrices

With a matrix group $G = \langle M_1, \ldots, M_k \rangle \leq \text{GL}_d(q)$ we can

- multiply, invert, compare, power up matrices
- execute straight line programs on matrices
- determine the order of a matrix $M$, i.e. $\min\{n \in \mathbb{N} \mid M^n = 1\}$
- find invariant subspaces $0 < W \leq \mathbb{F}^{1 \times d}$ with $W_g \subseteq W$ for all $g \in G$ or prove irreducibility: “MEATAXE”
- create (pseudo-) random elements
- act with matrices on vectors or on subspaces → gives homomorphisms to permutation groups
Homomorphisms

Try reduction: For $G = \langle M_1, \ldots, M_k \rangle \leq \text{GL}_d(q)$ find a homomorphism $\varphi : G \rightarrow H$ which is

- explicitly computable
- onto some group $H = \langle \varphi(M_1), \ldots, \varphi(M_k) \rangle$ which is “easier to handle”

Assume we can solve the word problem in $H$.
Set $N := \text{ker}(\varphi)$. Then:

- create a (pseudo-) random element $g$ in $G$
- map $g$ to $H$ via $\varphi$
- express $\varphi(g)$ as an SLP $S$ in $\varphi(M_1), \ldots, \varphi(M_k)$
- execute $S$ on $M_1, \ldots, M_k$, get $g' \in G$ s.t. $\varphi(g) = \varphi(g')$

$\implies g^{-1} \cdot g' \in N$

$\implies$ this creates a (pseudo-) random element in $N$
Composition trees

Produce generators of \( N := \ker(\varphi) \) and recognise. Assume that the word problem is solved in \( H \) and \( N \).

What does this help for \( G \)?

- \(|G| = |H| \cdot |N|\)
- \( G \) has a subgroup \( N \) and a factor group \( H \)
- we can solve the word problem in \( G \)!
Get the recursion going...

Choose as “nice generators” $M'_1, \ldots, M'_k$ for $G$:
- preimages under $\varphi$ of the nice generators of $H$ plus
- the nice generators of $N$

Given $g \in G$, find an SLP $S$ expressing $g$ in the $M'_i$:
- map $g$ via $\varphi$ to $\varphi(g) \in H$
- express $\varphi(g)$ as SLP $S'$ in the nice gens of $H$
- execute $S'$ on the preimages, get $g'$
- express $g'^{-1} \cdot g \in N$ as SLP $S''$ in $N$
- put together $S$ from $S'$ and $S''$ plus one multiplication
A Composition Tree

Upward arrows: monomorphisms
Downward arrows: epimorphisms
Low index

Assume:
- $G$ has a maximal subgroup $K$ of low index
- $G$ acts irreducibly
- $K$ leaves a subspace $0 < W < \mathbb{F}_q^{1 \times d}$ invariant

Try to find a homomorphism in the following way:
- create random elements, hope they generate $K$
- find an invariant subspace for these elements
- calculate its orbit under the action of $G$
- find a homomorphism onto a permutation group $H$

This works amazingly well!

Unfortunately, it is not yet analysed to be polynomial-time!
# Aschbacher’s Theorem

Aschbacher classified the maximal subgroups of \( \text{GL}_d(q) \).

## Theorem (Aschbacher, 1984)

If \( G < \text{GL}_d(q) \) then it falls under at least one of:

- **C1** \( G \) leaves invariant a subspace \( 0 < W < \mathbb{F}^{1 \times d}_q \)
- **C2** \( G \) preserves a decomposition \( \mathbb{F}^{1 \times d}_q \cong V_1 \oplus \cdots \oplus V_j \)
- ... 
- **C4** \( G \) preserves a decomposition \( \mathbb{F}^{1 \times d}_q \cong V_1 \otimes V_2 \)
- ... 
- **C8** \( G \) contains a “classical group” like \( \text{SL}_d(q) \) or \( \text{Sp}_d(q) \)
- **C9** \( G \) is a quasi-simple group

All classes C1 to C7 are defined “geometrically” and promise some kind of homomorphism or “simplification”.

C8 and C9 produce leaves in the composition tree.
Problem children

The leaves are a problem: Need representation theory.

Classify: Irred. modular representations of finite groups.

This is ongoing research, but there are many results.

We try to

- recognise the group for example by looking at distribution of element orders of random elements
- use collected data about representations or
- use collected data about subgroups
- directly solve the word problem.
A GAP package recog

Already there:

- a completely working framework for composition trees
- documentation of it
- a framework to administrate methods to find homomorphisms or leaves
- the infrastructure for SLPs, matrix handling, etc.
- background algorithms for orbits, MEATAXE, etc.
- handling of permutation groups in our framework
- homomorphisms using invariant subspaces
- a low index procedure (without analysis)
- methods to handle C1, C2, C4, and C6
- recognition of classical groups (C8)
- recognition of simple groups by the two largest element orders (C9)
- a start of a database of hints for recognised leaves
A GAP package recog

Still missing:

- analysis of the low index procedure
- methods to handle C3, C5, and C7
- solving the word problem after recognition of a classical group
- more hints in the database of hints for recognised leaves
- verification procedures (presentations)
- better methods, maybe “orthogonal” to the Aschbacher classification
- a whole lot of documentation
Help is appreciated

Everybody is welcome to contribute.

We need ideas, code, and analysis.