Case study: Parallel orbit enumeration

Max Neunhöffer

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(joint work with Christopher Brown, Kevin Hammond, Vladimir Janjic, Steve Linton and Hans-Wolfgang Loidl)
Problem (Orbit enumeration)

Let \( a : X \times G \to X \) and \( x_0 \in X \). Determine the smallest subset \( \mathcal{O} \subseteq X \), such that \( x_0 \in \mathcal{O} \) and: for all \( x \in \mathcal{O} \) and all \( g \in G \) we have \( a(x, g) \in \mathcal{O} \).
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Basic Orbit Algorithm

\begin{itemize}
  \item \textbf{Input:} \( x_0 \in X, g_1, g_2, \ldots, g_k : X \to X \)
  \item \( T := \{ x_0 \} \) (a hash table)
  \item \( O := [x_0] \) (a list)
  \item \( i := 1 \)
  \item while \( i \leq \text{Length}(O) \) do
    \item for \( j \) from 1 to \( k \) do
      \item \( y := O[i] \cdot g_j \)
      \item if \( y \notin T \) then
        \begin{itemize}
          \item Add \( y \) to \( T \)
          \item Add \( y \) to the end of \( O \)
        \end{itemize}
      \item \( i := i + 1 \)
  \end{itemize}

return \( O \) (containing the orbit of \( x_0 \))
A worker gets a chunk of points from some hash server, applies all generators to all points in the chunk, and sends all results to the responsible hash server. A distribution function regulates who is responsible. A hash server stores and recognises points, and keeps track of work to do.
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Input:

- the set $G$ and the action function $a : X \times G \to X$,
- the number $h$ of hash servers and
- a distribution hash function $f : X \to \{1, \ldots, h\}$

while TRUE do

get a chunk $C$ of points

$R :=$ a list of length $h$ of empty lists

for all $x \in C$ do

for all $g \in G$ do

$y := x \cdot g$

append $y$ to $R[f(y)]$

for all $j \in \{1, \ldots, h\}$ do

schedule sending $R[j]$ to hash server $j$
A hash server

**Input:** a chunk size $s$

**initialise** a hash table $T$ and a work queue $Q$

**while** TRUE **do**

- get a chunk $C$ of points (usually from a worker)
- for all $x \in C$ do
  - if $x \notin T$ then
    - add $x$ to $T$ and append it to $Q$
    - if at least $s$ points in $Q$ are unscheduled **then**
      - schedule a chunk of size $s$ points
  - if there are unscheduled points in $Q$ **then**
    - schedule a chunk of size $< s$ points
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hpcgap/demo/parorbit/parallelorbit2.g
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Vladimir will talk about the distributed memory implementation.
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- Each hash server has an input queue.
- There is a global work queue to send work to the workers.
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**In general**: **Never use blocking calls for communication!**
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Theorem (A priori runtime estimate)

Let $w$ be the number of workers and $h$ be the number of hash servers. Then the runtime of our algorithm is approximately

$$\max\left\{ |G| \cdot |O|^{A}, |G| \cdot |O|^{L} \right\},$$

where $A$ is the number of $\text{ACT}$ operations a worker can do per sec. and $L$ is the number of $\text{LOOKUP}$ operations a hash server can do per sec.
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Results and timings (shared memory)

Number of workers vs Speedup

2 Hash Servers

Ideal Speedup