Generalisations of Small Cancellation Theory

Max Neunhöffer

joint work with Jeffrey Burdges, Stephen Linton, Richard Parker and Colva Roney-Dougal

NBSAN meeting St Andrews, 9 April 2013
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We view each edge as a pair of opposite directed edges: half-edges.
Each half-edge is labelled at the start vertex and along the half-edge.
The diagram boundary problem

Let $R$ be a finite set of cyclic words, called relators.
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**Problem (Diagram boundary problem)**

Algorithmically devise a procedure that decides for any cyclic word $w$, whether or not there is a diagram such that

- every *internal region* is labelled by a *relator*, and
- the *external boundary* is labelled by $w$. 
Diagrams and their problems

The diagram boundary problem

Max Neunhöffer (University of St Andrews)
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Rules for the labels

We label every **half-edge** with **two symbols**, 
- one for the **corner** to the right of where it starts, and 
- one for the **right hand side** of it:

![Diagram](image)

We now need **rules** for the **corner labels** and the **edge labels**.
Definition (Corner structures)

A **corner structure** is a set $S$ with a subset $S_+ \subset S$, such that $S_0 := S \cup \{0\}$ is a semigroup with $0$ and:

$$\text{if } xy \in S_+ \text{ for } x, y \in S, \text{ then } yx \in S_+.$$

The elements in $S_+$ are called **acceptors**.
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Lemma (Cyclicity)

Let $S$ be a corner structure, if $s_1 s_2 \cdots s_k \in S_+$, then all rotations $s_i s_{i+1} \cdots s_k s_1 s_2 \cdots s_{i-1} \in S_+$. 
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Vertex rules

The corner labels are from a corner structure $S$. 

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Vertex rules

The corner labels are from a corner structure $S$, a vertex is valid, if the clockwise product of its corner labels is an acceptor.
Examples of corner structures
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- Let $G$ be a group. Let $P := G$ and $P_+ := \{1\}$. 

Note: $rl = e$ and $lr = s$, cyclicity, "inverses", two idempotents.
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Definition (Edge alphabet)

An edge alphabet is a set $X$ with an involution $\overline{\cdot} : X \to X$. 

(This is actually a special case of a corner structure.)

Edge rules
The edge labels are from an edge alphabet, a pair of half-edges forming an edge with labels $X$ and $Y$ is valid, if $Y = \overline{X}$. (For the experts: This is a generalisation of the rules of van Kampen diagrams.)
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Let $S$ be a corner structure and $X$ be an edge alphabet.

**Definition (Set of relators)**

A set of relators $R$ is a finite set of cyclic alternating words in $S$ and $X$. 

**Definition (Valid diagram)**

Let $R$ be a set of relators in $S$ and $X$. A valid diagram is:

A finite plane graph with half-edge set $\hat{E}$ and a labelling function $\ell : \hat{E} \to S \times X$, $e \mapsto (\ell_S(e), \ell_X(e))$, such that

$\ell_S(e_1) \cdot \ell_S(e_2) \cdot \ell_S(e_3) \cdot \ldots \cdot \ell_S(e_k) \in S^+$ for every clockwise cyclic sequence $e_1, e_2, \ldots, e_k$ of half-edges leaving the same vertex,

$\ell_X(e) = \ell_X(e')$ for all edges $\{e, e'\}$ consisting of half-edges $e, e'$,

$(\ell_S(e_1), \ell_X(e_1), ..., \ell_S(e_k), \ell_X(e_k)) \not\equiv R$ for every clockwise cycle $(e_1, e_2, ..., e_k) \not\equiv$ of half-edges around an internal face.
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Let $\langle S; X \mid R \rangle$ be a presentation, that is:
- $S$ is a corner structure,
- $X$ is an edge alphabet and
- $R$ is a set of relators in $S$ and $X$.

**Problem (Diagram boundary problem)**

Algorithmically devise a procedure that decides for any cyclic alternating word $w$ in $S$ and $X$ whether or not there is a valid diagram such that the external face is labelled by $w$. 

**Problem (Isoperimetric inequality)**

Algorithmically find and prove a function $D: \mathbb{N} \rightarrow \mathbb{N}$, such that for every cyclic alternating word $w$ in $S$ and $X$ of length $2k$ that is the boundary label of a valid diagram, there is one with at most $D(k)$ internal faces.

If there is a linear $D$, we call $\langle S; X \mid R \rangle$ hyperbolic.
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With $K_6$ we can do rewrite systems, if no rewrite has an empty side:

\[
S = \begin{bmatrix}
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  s & . & s & . & . & . \\
  t & . & s & t & . & l \\
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  b & . & . & e & b & r \\
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$X = \{A, B, C, D, E, F, G, U\}$ (id$_X$ is $\bar{0}$)

This encodes $UABCG \rightarrow DEF$ using:

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$S$ accepts $st^* + eb^* + rt^* lb^*$ and all rotations.
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Other Applications

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You just have to choose the right corner structure and edge alphabet!
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Find “pieces”, and remove vertices of valency 1 and 2:
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Euler’s formula

The total sum of our combinatorial curvature is always $+1$. 
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**Combinatorial curvature:** We endowed
  - each vertex with $+1$ unit of combinatorial curvature,
  - each edge with $-1$ unit of combinatorial curvature and
  - each internal face with $+1$ unit of combinatorial curvature.

Euler's formula: The total sum of our combinatorial curvature is always $+1$. 

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Idea (Curvature redistribution — Officers)

We redistribute the curvature locally in a conservative way.
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In Phase 1 Tom moves the negative curvature to the vertices:

A vertex with valency $v \geq 3$ will now have $\frac{-1}{2} - \frac{1}{2} < 0$.

Faces still have $+1$, edges now have $0$. 
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An algorithmic approach

Curvature redistribution — Phase 1 of officer Tom

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Corner values for Tom

A corner value $c$ of Tom depends on two edges that are adjacent on a face.
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A corner value $c$ of Tom depends on **two edges** that are adjacent on a **face**. Tom moves $c$ units of curvature **from the face to the vertex**. The **default value** for $c$ is $1/6$ if the vertex can have **valency 3** and $1/4$ otherwise.
Tom — and officers in general — want to redistribute the curvature, such that for all permitted diagrams after redistribution

- every internal face has $< -\varepsilon$ curvature (for some explicit $\varepsilon > 0$),
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An algorithmic approach

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\[ 1 < n - F \cdot \varepsilon \implies F < \varepsilon^{-1} \cdot n \implies \text{hyperbolic} \]
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If $S \geq 0$ then there is a $j \in L$ such that for all $i \in \mathbb{N}$ the partial sum

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**Corollary**

Assume that there are $k \in \mathbb{N}$ and $\varepsilon \geq 0$ such that for all $j \in L$ there is an $i \leq k$ with $s_{j,i} < -\varepsilon$, then $S < -\varepsilon \cdot \ell/k$. 
Sunflower

To show that every internal face has curvature $< -\varepsilon$:

Use Goes Up and Stays Up on $\frac{L_1 + L_2}{2L} - c$. 
To show that every internal vertex has curvature $\leq 0$:

Use **Goes Up and Stays Up** on $c + \frac{1-v/2}{v} = c + \frac{2-v}{v}$. 

\[ \text{Diagram showing vertices and edges.} \]
To show that every internal vertex has curvature $\leq 0$:

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Do valency $v = 3$ first, if nothing found, increase $v$. 

**Diagram:**

- $c_1$
- $c_2$
- $c_3$
- $c_4$
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This **terminates**: higher valencies tend to be **negatively curved** anyway.
Overview over Tom analysis

What have we achieved?

If we did not find any bad sunflower or poppy, we have determined an explicit $\epsilon$, proved hyperbolicity, and can in principle solve the diagram boundary problem.

If we did find bad sunflowers or poppy, we can still improve our choices for the corner values (leads to difficult optimisation/linear program problems), forbid more diagrams (if possible) (need to show that every boundary is proved by a permitted one), or switch to a more powerful officer (with further sight or redistribution),...

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If $\langle S, X; R \rangle$ is not hyperbolic, this will not work.
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