1. Show that $\gamma_2(G) = G'$. Deduce that abelian groups are nilpotent.

**Solution:** By definition $\gamma_2(G) = [G, G]$. Thus $\gamma_2(G) = \langle [x, y] \mid x, y \in G \rangle = G'$. If $G$ is abelian, then $[x, y] = 1$ for all $x, y \in G$, so $\gamma_2(G) = G' = 1$. Hence $G$ is nilpotent (of class $\leq 1$).

2. Show that $Z(S_3) = 1$. Hence calculate the upper central series of $S_3$ and deduce that $S_3$ is not nilpotent.

Show that $\gamma_i(S_3) = A_3$ for all $i \geq 2$. [Hint: We have calculated $S_3^i$ previously and now know that $S_3$ is not nilpotent.]

Find a normal subgroup $N$ of $S_3$ such that $S_3/N$ and $N$ are both nilpotent.

**Solution:** Recall that all permutations with the same cycle structure are conjugate in $S_n$. Therefore a permutation lies in the centre of $S_3$ if and only if it is the only permutation of its cycle structure. Hence $Z(S_3) = 1$ (there are three permutations of cycle structure $(\alpha \beta)$ and two of cycle structure $(\alpha \beta \gamma)$).

This shows that $Z_i(S_3) = 1$. Suppose that $Z_i(S_3) = 1$. Then $Z_{i+1}(S_3) = Z_{i+1}(S_3)/Z_i(S_3) = Z(S_3/Z_i(S_3)) = Z(S_3) = 1$. Hence, by induction, $Z_i(S_3) = 1$ for all $i$. Since $Z_i(S_3) < S_3$ for all $i$, we deduce that $S_3$ is not nilpotent.

Now $S_3^i = A_3$, by Question 2(i) on Problem Sheet VI. Hence $\gamma_2(S_3) = S_3^i = A_3$. Now $A_3$ is of order 3, so has no proper non-trivial subgroups. Hence for $i \geq 2$, either $\gamma_i(S_3) = A_3$ or $\gamma_i(S_3) = 1$. But $S_3$ is not nilpotent, so $\gamma_i(S_3) \neq 1$ for all $i$. Hence $\gamma_i(S_3) = A_3$ for all $i \geq 2$.

Let $N = A_3 \leq S_3$. Then $S_3/N \cong C_2$ and $N \cong C_3$, so these are both abelian and hence nilpotent. (Thus we have an example of a non-nilpotent group $G$ with normal subgroup $N$ such that $G/N$ and $N$ are nilpotent.)

3. Show that $Z(G \times H) = Z(G) \times Z(H)$.

Show, by induction on $i$, that $Z_i(G \times H) = Z_i(G) \times Z_i(H)$ for all $i$.

Deduce that a direct product of a finite number of nilpotent groups is nilpotent.

**Solution:** Let $(x, y) \in Z(G \times H)$. Then for $g \in G$ and $h \in H$, it follows that $(x, y)(g, h) = (g, h)(x, y)$. That is, $(xg, yh) = (gx, hy)$. Hence $xg = gx$ for all $g \in G$, and $yh = hy$ for all $h \in H$. Therefore $x \in Z(G)$ and $y \in Z(H)$, so $Z_i(G \times H) \leq Z(G) \times Z(H)$.

Conversely, if $(x, y) \in Z(G) \times Z(H)$; that is, $x \in Z(G)$ and $y \in Z(H)$, then

$$(x, y)(g, h) = (xg, yh) = (gx, hy) = (g, h)(x, y)$$

so $(x, y) \in Z(G \times H)$. This shows that $Z(G) \times Z(H) \leq Z(G \times H)$. The equality now follows.
For the next step, induct on $i$. If $i = 0$, then $Z_0(G \times H) = \{(1, 1)\} = 1 \times 1 = Z_0(G) \times Z_0(H)$, so the result holds. Suppose as an inductive hypothesis that $Z_i(G \times H) = Z_i(G) \times Z_i(H)$. Then

$$\frac{G \times H}{Z_i(G \times H)} = \frac{G \times H}{Z_i(G) \times Z_i(H)}.$$ 

The map $\phi$ that sends $(Z_i(G) \times Z_i(H))(x, y)$ to $(Z_i(G)x, Z_i(H)y)$ is an isomorphism:

$$\phi: \frac{G \times H}{Z_i(G) \times Z_i(H)} \to \frac{G}{Z_i(G)} \times \frac{H}{Z_i(H)}.$$ 

(This works whenever $M \trianglelefteq G$ and $N \trianglelefteq G$, for then $(G \times H)/(M \times N) \cong G/M \times H/N$ via a similar isomorphism.) This isomorphism $\phi$ maps the centre of the group on the left-hand side to the centre of the group on the right-hand side. Hence

$$\left(\frac{Z_{i+1}(G \times H)}{Z_i(G \times H)}\right) \phi = \left(\frac{Z_i \left(\frac{G \times H}{Z_i(G \times H)}\right)}{Z_i(G \times H)}\right) \phi$$

$$= \left(\frac{Z_i \left(\frac{G \times H}{Z_i(G) \times Z_i(H)}\right)}{Z_i(G) \times Z_i(H)}\right) \phi$$

$$= Z_i(\frac{G/Z_i(G) \times H/Z_i(H)})$$

$$= Z_i(\frac{G/Z_i(G)}{Z_i(H)}) \times Z_i(H/Z_i(G)) \quad \text{from the first part}$$

$$= Z_{i+1}(G)/Z_i(G) \times Z_{i+1}(H)/Z_i(H) \quad \text{by definition}$$

$$= \left(\frac{Z_{i+1}(G) \times Z_{i+1}(H)}{Z_i(G) \times Z_i(H)}\right) \phi$$

with the last step being the definition of $\phi$. Since $\phi$ is a bijection,

$$\frac{Z_{i+1}(G \times H)}{Z_i(G \times H)} = \frac{Z_{i+1}(G) \times Z_{i+1}(H)}{Z_i(G \times H)}$$

and the Correspondence Theorem yields $Z_{i+1}(G \times H) = Z_{i+1}(G) \times Z_{i+1}(H)$, which completes the induction.

Let $G_1, G_2, \ldots, G_n$ be nilpotent groups. Then there exist $c_i$ such that $Z_{c_i}(G_i) = G_i$. Choose $c$ to be the largest of all the $c_i$. Then $Z_c(G_i) = G_i$ for $i = 1, 2, \ldots, n$. By the previous result, we see that

$$Z_c(G_1 \times G_2 \times \cdots \times G_n) = Z_c(G_1) \times Z_c(G_2) \times \cdots \times Z_c(G_n)$$

$$= G_1 \times G_2 \times \cdots \times G_n,$$

and hence $G_1 \times G_2 \times \cdots \times G_n$ is nilpotent.

4. Let $G$ be a finite elementary abelian $p$-group. Show that $\Phi(G) = 1$.

Solution: Let $G = C_p \times C_p \times \cdots \times C_p$ ($d$ times, for some $d$). Then

$$M = M_i = C_p \times \cdots \times C_p \times 1 \times C_p \times \cdots \times C_p$$

(where the 1 occurs in the $i$th entry) is a subgroup of $G$ of index $p$. If $H$ is a subgroup of $G$ such that $M \trianglelefteq H \trianglelefteq G$, then $|G : H| = [H : M] = |G : M| = p$, so as $p$ is prime, either $H = G$ or $H = M$. Hence $M$ is a maximal subgroup of $G$. Clearly

$$\bigcap_{i=1}^d M_i = 1$$
and this is the intersection of just some of the maximal subgroups of $G$. Hence

$$\Phi(G) = \bigcap_{M \text{ maximal in } G} M \leq \bigcap_{i=1}^d M_i = 1.$$  

5. Let $G$ be a finite $p$-group. 
If $M$ is a maximal subgroup of $G$, show that $|G : M| = p$. [Hint: $G$ is nilpotent, so $M \trianglelefteq G$.] 
Deduce that $G^pG' \subseteq \Phi(G)$. 
Use the previous question to show that $\Phi(G) = G^pG'$. 

Solution: Since $G$ is a finite $p$-group, it is nilpotent (Example 7.6). Let $M$ be a maximal subgroup of $G$. Then $M \trianglelefteq G$ (Lemma 7.15), and $G/M$ possesses no non-trivial proper subgroups (by the Correspondence Theorem). Therefore $G/M$ is cyclic of prime order, so $|G : M| = p$. 
If $x \in G$, then $(Mx)^p = M1$, so $x^p \in M$. Hence

$$x^p \in \bigcap_{M \text{ maximal in } G} M = \Phi(G) \quad \text{for all } x \in G.$$  

We deduce that $G^p = \langle x^p \mid x \in G \rangle \subseteq \Phi(G)$. We have already observed that $G' \trianglelefteq \Phi(G)$ (see Theorem 7.18), so $G^pG' \subseteq \Phi(G)$.

Let $N = G^pG'$. This is a product of two normal subgroups of $G$, so $N \trianglelefteq G$. Now $G/N$ is abelian (since $G' \trianglelefteq N$) and if $x \in G$, then

$$(Nx)^p = Nx^p = N1$$

(since $x^p \in G^p \subseteq N$). Hence $G/N$ is an elementary abelian $p$-group. It is therefore a direct product of a number of copies of $C_p$. The previous question now gives $\Phi(G/N) = 1$. Hence there is a collection $M_1, M_2, \ldots, M_k$ of subgroups of $G$ containing $N$ such that $M_i/N$ is a maximal subgroup of $G/N$ and $\bigcap_{i=1}^k (M_i/N) = 1$. By the Correspondence Theorem, $M_i$ is a maximal subgroup of $G$ and

$$\bigcap_{i=1}^k M_i = N.$$  

Hence

$$\Phi(G) = \bigcap_{M \text{ maximal in } G} M \leq \bigcap_{i=1}^k M_i = N = G^pG'.$$  

Taken together with the previous inclusion, $\Phi(G) = G^pG'$. 
Now as $G/\Phi(G)$ is an elementary abelian $p$-group, it is a direct product of $d$ copies of the cyclic group $C_p$ (for some $d$). Choose $x_1, x_2, \ldots, x_d \in G$ such that

$$\Phi(G)x_1, \Phi(G)x_2, \ldots, \Phi(G)x_d$$

are the generators of these $d$ direct factors. If $g \in G$, then

$$\Phi(G)g = \Phi(G)x_1^{e_1}x_2^{e_2} \cdots x_d^{e_d}$$

where $e_1, e_2, \ldots, e_d$ are non-negative integers.
for some \( e_i \in \{0, 1, \ldots, p-1\} \), so \( g = yx_1^{e_1}x_2^{e_2} \ldots x_d^{e_d} \) where \( y \in \Phi(G) \). Hence

\[
G = \langle x_1, x_2, \ldots, x_d, \Phi(G) \rangle.
\]

Suppose that \( x_1, x_2, \ldots, x_d \) do not generate \( G \). Then \( \langle x_1, x_2, \ldots, x_d \rangle \) is a proper subgroup of \( G \), so there exists a maximal subgroup \( M \) such that

\[
\langle x_1, x_2, \ldots, x_d \rangle \leq M < G.
\]

Then \( x_1, x_2, \ldots, x_d \in M \) while, by definition, \( \Phi(G) \leq M \). Hence

\[
G = \langle x_1, x_2, \ldots, x_d, \Phi(G) \rangle \leq M < G,
\]

a contradiction. So \( x_1, x_2, \ldots, x_d \) generate \( G \). This shows that if \( G/\Phi(G) \) is a direct product of \( d \) copies of \( C_p \), then \( G \) can be generated by \( d \) elements.

On the other hand, if \( G \) can be generated by \( d \) elements, then so can every quotient. A direct product of more than \( d \) copies of \( C_p \) cannot be generated by \( d \) elements, so the number of copies of \( C_p \) appearing in the direct product for \( G/\Phi(G) \) is at most \( d \).

Putting the above together we deduce that \( G \) can be generated by precisely \( d \) elements a and no fewer b if and only if \( G/\Phi(G) \) is a direct product of \( d \) copies of the cyclic group \( C_p \) of order \( p \).

6. Let \( G \) be a nilpotent group with lower central series

\[
G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_c(G) > \gamma_{c+1}(G) = 1.
\]

Suppose \( N \) is a non-trivial normal subgroup of \( G \). Choose \( i \) to be the largest positive integer such that \( N \cap \gamma_i(G) \neq 1 \). Show that \( [N \cap \gamma_i(G), G] = 1 \).

Deduce that \( N \cap Z(G) \neq 1 \).

**Solution:** \( N \neq 1 \), so \( N \cap \gamma_1(G) = N \cap G = N \neq 1 \). Hence we may choose \( i \) to be the largest positive integer such that \( N \cap \gamma_i(G) \neq 1 \). Then

\[
[N \cap \gamma_i(G), G] = [\gamma_i(G), G] = \gamma_{i+1}(G)
\]

while

\[
[N \cap \gamma_i(G), G] \leq [N, G] \leq N
\]

since \( N \leq G \) (if \( x \in N \) and \( g \in G \), then \([x, g] = x^{-1}xg \in N \)). Hence

\[
[N \cap \gamma_i(G), G] \leq N \cap \gamma_{i+1}(G) = 1
\]

by the hypothesis that \( i \) is largest with the given property.

Hence \( N \cap \gamma_i(G) \leq Z(G) \) since \([x, g] = 1\) for all \( x \in N \cap \gamma_i(G) \) and all \( g \in G \). Therefore

\[
1 \neq N \cap \gamma_i(G) \leq N \cap Z(G)
\]

so \( N \cap Z(G) \neq 1 \).