Chapter 7

Nilpotent Groups

Recall the commutator is given by

\[ [x, y] = x^{-1}y^{-1}xy. \]

**Definition 7.1** Let \( A \) and \( B \) be subgroups of a group \( G \). Define the commutator subgroup \([A, B]\) by

\[ [A, B] = \langle [a, b] \mid a \in A, b \in B \rangle, \]

the subgroup generated by all commutators \([a, b]\) with \( a \in A \) and \( b \in B \).

In this notation, the derived series is given recursively by \( G^{(i+1)} = [G^{(i)}, G^{(i)}] \) for all \( i \).

**Definition 7.2** The lower central series \((\gamma_i(G))\) (for \( i \geq 1 \)) is the chain of subgroups of the group \( G \) defined by

\[ \gamma_1(G) = G \]

and

\[ \gamma_{i+1}(G) = [\gamma_i(G), G] \quad \text{for} \quad i \geq 1. \]

**Definition 7.3** A group \( G \) is nilpotent if \( \gamma_{c+1}(G) = 1 \) for some \( c \). The least such \( c \) is the nilpotency class of \( G \).

It is easy to see that \( G^{(i)} \leq \gamma_{i+1}(G) \) for all \( i \) (by induction on \( i \)). Thus if \( G \) is nilpotent, then \( G \) is soluble. Note also that \( \gamma_2(G) = G' \).

**Lemma 7.4** (i) If \( H \) is a subgroup of \( G \), then \( \gamma_i(H) \leq \gamma_i(G) \) for all \( i \).

(ii) If \( \phi: G \to K \) is a surjective homomorphism, then \( \gamma_i(G)\phi = \gamma_i(K) \) for all \( i \).
(iii) \( \gamma_i(G) \) is a characteristic subgroup of \( G \) for all \( i \).

(iv) The lower central series of \( G \) is a chain of subgroups

\[
G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \gamma_3(G) \trianglerighteq \cdots.
\]

**Proof:** (i) Induct on \( i \). Note that \( \gamma_1(H) = H \trianglelefteq G = \gamma_1(G) \). If we assume that \( \gamma_i(H) \trianglelefteq \gamma_i(G) \), then this together with \( H \trianglelefteq G \) gives

\[
[\gamma_i(H), H] \trianglelefteq [\gamma_i(G), G]
\]

so \( \gamma_{i+1}(H) \trianglelefteq \gamma_{i+1}(G) \).

(ii) Induct on \( i \). Note that \( \gamma_1(G) \phi = G \phi = K = \gamma_1(K) \). Suppose \( \gamma_i(G) \phi = \gamma_i(K) \). If \( x \in \gamma_i(G) \) and \( y \in G \), then

\[
[x, y] \phi = [x \phi, y \phi] \in [\gamma_i(G) \phi, G \phi] = [\gamma_i(K), K] = \gamma_{i+1}(K),
\]

so \( \gamma_{i+1}(G) \phi = [\gamma_i(G), G] \phi \trianglelefteq \gamma_{i+1}(K) \).

On the other hand, if \( a \in \gamma_i(K) \) and \( b \in K \), then \( a = x \phi \) and \( b = y \phi \) for some \( x \in \gamma_i(G) \) and \( y \in G \). So

\[
[a, b] = [x \phi, y \phi] = [x, y] \phi \in [\gamma_i(G), G] \phi = \gamma_{i+1}(G) \phi.
\]

Thus \( \gamma_{i+1}(K) = [\gamma_i(K), K] \trianglelefteq \gamma_{i+1}(G) \phi \).

(iii) If \( \phi \) is an automorphism of \( G \), then \( \phi : G \to G \) is a surjective homomorphism, so from (ii)

\[
\gamma_i(G) \phi = \gamma_i(G).
\]

Thus \( \gamma_i(G) \) char \( G \).

(iv) From (iii), \( \gamma_i(G) \leq G \). Hence if \( x \in \gamma_i(G) \) and \( y \in G \), then

\[
[x, y] = x^{-1}x^y \in \gamma_i(G).
\]

Hence

\[
\gamma_{i+1}(G) = [\gamma_i(G), G] \leq \gamma_i(G) \quad \text{for all } i.
\]

\[\square\]

We deduce two consequences immediately:

**Lemma 7.5** Subgroups and homomorphic images of nilpotent groups are themselves nilpotent.

**Proof:** Let \( \gamma_{c+1}(G) = 1 \) and \( H \leq G \). Then by Lemma 7.4(i), \( \gamma_{c+1}(H) \leq \gamma_{c+1}(G) = 1 \), so \( \gamma_{c+1}(H) = 1 \) and \( H \) is nilpotent.

Let \( \phi : G \to K \) be a surjective homomorphism. Then Lemma 7.4(ii) gives \( \gamma_{c+1}(K) = \gamma_{c+1}(G) \phi = 1 \phi = 1 \), so \( K \) is nilpotent. \[\square\]
Note, however, that
\[ N \trianglelefteq G, \ G/N \text{ and } N \text{ nilpotent } \nRightarrow G \text{ nilpotent}. \]
In this way, nilpotent groups are different to soluble groups.

**Example 7.6** Finite \( p \)-groups are nilpotent.

**Proof:** Let \( G \) be a finite \( p \)-group, say \( |G| = p^n \). We proceed by induction on \( |G| \). If \( |G| = 1 \), then \( \gamma_1(G) = G = 1 \) so \( G \) is nilpotent.

Now suppose \( |G| > 1 \). Apply Corollary 2.41: \( Z(G) \neq 1 \). Consider the quotient group \( G/Z(G) \). This is a \( p \)-group of order smaller than \( G \), so by induction it is nilpotent, say
\[
\gamma_{c+1}(G/Z(G)) = 1.
\]
Let \( \pi: G \to G/Z(G) \) be the natural homomorphism. Then by Lemma 7.4(ii),
\[
\gamma_{c+1}(G)\pi = \gamma_{c+1}(G/Z(G)) = 1,
\]
so \( \gamma_{c+1}(G) \leq \ker \pi = Z(G) \). Thus
\[
\gamma_{c+2}(G) = [\gamma_{c+1}(G), G] \leq [Z(G), G] = 1,
\]
so \( G \) is nilpotent. \( \square \)

The example illustrates that the centre has a significant role in the study of nilpotent groups. We make two further definitions:

**Definition 7.7** The **upper central series** of \( G \), denoted \( (Z_i(G)) \) for \( i \geq 0 \), is the chain of subgroups defined by
\[
Z_0(G) = 1; \quad Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)) \quad \text{for } i \geq 0.
\]
Suppose that \( Z_i(G) \trianglelefteq G \). Then \( Z(G/Z_i(G)) \) is a normal subgroup of \( G/Z_i(G) \), so corresponds to a normal subgroup \( Z_{i+1}(G) \) of \( G \) containing \( Z_i(G) \) by the Correspondence Theorem. In this way we define a chain of subgroups
\[
1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots,
\]
each of which is normal in \( G \). Here \( Z_1(G) = Z(G) \).

**Definition 7.8** A **central series** for a group \( G \) is a chain of subgroups
\[
G = G_0 \geq G_1 \geq \cdots \geq G_n = 1
\]
such that \( G_i \) is a normal subgroup of \( G \) and \( G_{i-1}/G_i \leq Z(G/G_i) \) for all \( i \).
Lemma 7.9 Let
\[ G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1 \]
be a central series for \( G \). Then for all \( i \):
\[ \gamma_{i+1}(G) \leq G_i \quad \text{and} \quad Z_i(G) \triangleright G_{n-i}. \]

**Proof:** First observe that \( \gamma_1(G) = G = G_0 \). Suppose that \( \gamma_i(G) \leq G_{i-1} \) for some \( i \). If \( x \in \gamma_i(G) \) and \( y \in G \), then
\[ G_i x \in G_{i-1}/G_i \leq Z(G/G_i), \]
so \( G_i x \) commutes with \( G_i y \). Therefore
\[ G_i [x, y] = (G_i x)^{-1}(G_i y)^{-1}(G_i x)(G_i y) = G_i, \]
so \( [x, y] \in G_i \). Hence
\[ \gamma_{i+1}(G) = [\gamma_i(G), G] \leq G_i. \]
Thus, by induction, the first inclusion holds.

Now, \( Z_0(G) = 1 = G_n \). Suppose that \( Z_i(G) \triangleright G_{n-i} \). Since \( (G_i) \) is a central series for \( G \),
\[ G_{n-i-1}/G_{n-i} \leq Z(G/G_{n-i}). \]
Thus if \( x \in G_{n-i-1} \) and \( y \in G \), then
\[ G_{n-i} x \text{ and } G_{n-i} y \text{ commute; i.e., } [x, y] \in G_{n-i}. \]
Hence \( [x, y] \in Z_i(G) \), so \( Z_i(G)x \) and \( Z_i(G)y \) commute. Since \( y \) is an arbitrary element of \( G \), we deduce that
\[ Z_i(G)x \in Z(G/Z_i(G)) = Z_{i+1}(G)/Z_i(G) \]
for all \( x \in G_{n-i-1} \). Thus \( G_{n-i-1} \leq Z_{i+1}(G) \) and the second inclusion holds by induction. \( \square \)

We have now established the link between a general central series and the behaviour of the lower and the upper central series.

**Theorem 7.10** The following conditions are equivalent for a group \( G \):

(i) \( \gamma_{c+1}(G) = 1 \) for some \( c \);

(ii) \( Z_c(G) = G \) for some \( c \);

(iii) \( G \) has a central series.

Thus these are equivalent conditions for a group to be nilpotent.
Proof: If $G$ has a central series $(G_i)$ of length $n$, then Lemma 7.9 gives
\[ \gamma_{n+1}(G) \leq G_n = 1 \quad \text{and} \quad Z_n(G) \geq G_0 = G. \]
Hence (iii) implies both (i) and (ii).

If $Z_c(G) = G$, then
\[ G = Z_c(G) \geq Z_{c-1}(G) \geq \cdots \geq Z_1(G) \geq Z_0(G) = 1 \]
is a central series for $G$ (as $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$). Thus (ii) implies (iii).

If $\gamma_{c+1}(G) = 1$, then
\[ G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_{c+1}(G) = 1 \]
is a central series for $G$. (For if $x \in \gamma_{i-1}(G)$ and $y \in G$, then $[x, y] \in \gamma_i(G)$, so $\gamma_i(G)x$ and $\gamma_i(G)y$ commute for all such $x$ and $y$; thus $\gamma_{i-1}(G)/\gamma_i(G) \leq Z(G/\gamma_i(G)).$) Hence (i) implies (iii).

Further examination of this proof and Lemma 7.9 shows that
\[ \gamma_{c+1}(G) = 1 \quad \text{if and only if} \quad Z_c(G) = G. \]
Thus for a nilpotent group, the lower central series and the upper central series have the same length.

Our next goal is to develop further equivalent conditions for finite groups to be nilpotent.

**Proposition 7.11** Let $G$ be a nilpotent group. Then every proper subgroup of $G$ is properly contained in its normaliser:
\[ H < N_G(H) \quad \text{whenever} \ H < G. \]

Proof: Let
\[ G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_{c+1}(G) = 1 \]
be the lower central series. Then $\gamma_{c+1}(G) \leq H$ but $\gamma_1(G) \not\subseteq H$. Choose $i$ as small as possible so that $\gamma_i(G) \leq H$. Then $\gamma_{i-1}(G) \not\subseteq H$. Now
\[ [\gamma_{i-1}(G), H] \leq [\gamma_{i-1}(G), G] = \gamma_i(G) \leq H, \]
so
\[ x^{-1}hx^{-1} = [x, h^{-1}] \in H \quad \text{for} \ x \in \gamma_{i-1}(G) \text{ and } h \in H. \]
Therefore
\[ x^{-1}hx \in H \quad \text{for} \ x \in \gamma_{i-1}(G) \text{ and } h \in H. \]
We deduce that $H^x = H$ for all $x \in \gamma_{i-1}(G)$, so that $\gamma_{i-1}(G) \leq N_G(H)$. Therefore, since $\gamma_{i-1}(G) \not\subseteq H$, we deduce $N_G(H) > H$. \(\square\)
Let us now analyse how nilpotency affects the Sylow subgroups of a finite group. This links into the previous proposition via the following lemma.

**Lemma 7.12** Let $G$ be a finite group and let $P$ be a Sylow $p$-subgroup of $G$ for some prime $p$. Then

$$N_G(N_G(P)) = N_G(P).$$

**Proof:** Let $H = N_G(P)$. Then $P \trianglelefteq H$, so $P$ is the unique Sylow $p$-subgroup of $H$. (Note that as it is a Sylow $p$-subgroup of $G$ and $P \trianglelefteq H$, it is also a Sylow $p$-subgroup of $H$, as it must have the largest possible order for a $p$-subgroup of $H$.) Let $g \in N_G(H)$. Then

$$P^g \trianglelefteq H^g = H,$$

so $P^g$ is also a Sylow $p$-subgroup of $H$ and we deduce $P^g = P$; that is, $g \in N_G(P) = H$. Thus $N_G(H) \trianglelefteq H$, so we deduce

$$N_G(H) = H,$$

as required.

We can now characterise finite nilpotent groups as being built from $p$-groups in the most simple way.

**Theorem 7.13** Let $G$ be a finite group. The following conditions on $G$ are equivalent:

(i) $G$ is nilpotent;

(ii) every Sylow subgroup of $G$ is normal;

(iii) $G$ is a direct product of $p$-groups (for various primes $p$).

**Proof:** (i) $\Rightarrow$ (ii): Let $G$ be nilpotent and $P$ be a Sylow $p$-subgroup of $G$ (for some prime $p$). Let $H = N_G(P)$. By Lemma 7.12, $N_G(H) = H$. Hence, by Proposition 7.11, $H = G$. That is, $N_G(P) = G$ and so $P \trianglelefteq G$.

(ii) $\Rightarrow$ (iii): Let $p_1, p_2, \ldots, p_k$ be the distinct prime factors of $|G|$, say

$$|G| = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

and assume that $G$ has a normal Sylow $p_i$-subgroup $P_i$ for $i = 1, 2, \ldots, k$. 

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Claim: $P_1P_2\ldots P_j \cong P_1 \times P_2 \times \cdots \times P_j$ for all $j$.

Certainly this claim holds for $j = 1$. Assume it holds for some $j$, and consider $N = P_1P_2\ldots P_j \cong P_1 \times \cdots \times P_j \unlhd G$ and $P_{j+1} \unlhd G$. Then $|N|$ is coprime to $|P_{j+1}|$. Hence $N \cap P_{j+1} = 1$ and therefore $NP_{j+1}$ satisfies the conditions to be an (internal) direct product. Thus

$$NP_{j+1} \cong N \times P_{j+1} \cong P_1 \times P_2 \times \cdots \times P_j \times P_{j+1},$$

and by induction the claim holds.

In particular, note

$$|P_1P_2\ldots P_k| = |P_1 \times P_2 \times \cdots \times P_k| = |P_1| \cdot |P_2| \cdot \ldots \cdot |P_k| = |G|,$$

so

$$G = P_1P_2\ldots P_k \cong P_1 \times P_2 \times \cdots \times P_k.$$

(iii) $\Rightarrow$ (i): Suppose $G = P_1 \times P_2 \times \cdots \times P_k$, a direct product of non-trivial $p$-groups. Then

$$Z(G) = Z(P_1) \times Z(P_2) \times \cdots \times Z(P_k) \neq 1$$

(by Corollary 2.41). Then

$$G/Z(G) = P_1/Z(P_1) \times P_2/Z(P_2) \times \cdots \times P_k/Z(P_k)$$

is a direct product of $p$-groups of smaller order. By induction, $G/Z(G)$ is nilpotent, say $\gamma_c(G/Z(G)) = 1$. Now apply Lemma 7.4(ii) to the natural map $\pi: G \to G/Z(G)$ to see that $\gamma_c(G)\pi = \gamma_c(G/Z(G)) = 1$. Thus $\gamma_c(G) \leqslant \ker \pi = Z(G)$ and hence

$$\gamma_{c+1}(G) = [\gamma_c(G), G] \leqslant [Z(G), G] = 1.$$ 

Therefore $G$ is nilpotent. \[\square\]

This tells us that the study of finite nilpotent groups reduces to understanding $p$-groups. We finish by introducing the Frattini subgroup, which is of significance in many parts of group theory.

**Definition 7.14** A maximal subgroup of a group $G$ is a subgroup $M < G$ such that there is no subgroup $H$ with $M < H < G$.

Thus a maximal subgroup is a proper subgroup which is largest amongst the proper subgroups.

If $G$ is a nilpotent group, then Proposition 7.11 tells us that

$$M < N_G(M) \leqslant G,$$

for any maximal subgroup $M$ of $G$. The maximality of $M$ forces $N_G(M) = G$; that is, $M \not\leqslant G$. Thus:

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**Lemma 7.15** Let $G$ be a nilpotent group. Then every maximal subgroup of $G$ is normal in $G$. □

**Definition 7.16** The *Frattini subgroup* $\Phi(G)$ of a group $G$ is the intersection of all its maximal subgroups:

$$\Phi(G) = \bigcap_{M \text{ maximal in } G} M.$$ 

(If $G$ is an (infinite) group with no maximal subgroups, then $\Phi(G) = G$.)

If we apply an automorphism to a maximal subgroup, we map it to another maximal subgroup. Hence the automorphism group permutes the maximal subgroups of $G$.

**Lemma 7.17** If $G$ is a group, then the Frattini subgroup $\Phi(G)$ is a characteristic subgroup of $G$. □

Our final theorem characterising nilpotent finite groups is:

**Theorem 7.18** Let $G$ be a finite group. The following are equivalent:

(i) $G$ is nilpotent;

(ii) $H < N_G(H)$ for all $H < G$;

(iii) every maximal subgroup of $G$ is normal;

(iv) $\Phi(G) \geq G'$;

(v) every Sylow subgroup of $G$ is normal;

(vi) $G$ is a direct product of $p$-groups.

**Proof:** We have already proved that (i) $\Rightarrow$ (ii) (Proposition 7.11), (ii) $\Rightarrow$ (iii) (see the proof of Lemma 7.15) and (v) $\Rightarrow$ (vi) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (iv): Let $M$ be a maximal subgroup of $G$. By assumption, $M \trianglelefteq G$. Since $M$ is maximal, the Correspondence Theorem tells us that $G/M$ has no non-trivial proper subgroups. It follows that $G/M$ is cyclic and so is abelian. Lemma 6.16 gives

$$G' \leq M.$$ 

Hence

$$G' \leq \bigcap_{M \text{ max } G} M = \Phi(G).$$
(iv) ⇒ (v): Let $P$ be a Sylow $p$-subgroup of $G$ and let $N = P \Phi(G)$ (which is a subgroup of $G$, since $\Phi(G) \trianglelefteq G$ by Lemma 7.17). Let $x \in N$ and $g \in G$. Then

$$x^{-1}x^g = [x, g] \in G' \leq \Phi(G) \leq N.$$  

Hence $x^g \in N$ for all $x \in N$ and $g \in G$, so $N \trianglelefteq G$. Now $P$ is a Sylow $p$-subgroup of $N$ (since it is the largest possible $p$-subgroup of $G$, so is certainly largest amongst $p$-subgroups of $N$). Apply the Frattini Argument (Lemma 6.35):

$$G = N_G(P) N$$

$$= N_G(P) P \Phi(G)$$

$$= N_G(P) \Phi(G) \quad (\text{as } P \leq N_G(P)).$$

From this we deduce that $G = N_G(P)$: for suppose $N_G(P) \neq G$. Then $N_G(P) \leq M < G$ for some maximal subgroup $M$ of $G$. By definition, $\Phi(G) \leq M$, so

$$N_G(P) \Phi(G) \leq M < G,$$

a contradiction. Hence $N_G(P) = G$ and so $P \trianglelefteq G$.

This completes all remaining stages in the proof. 

\[ \Box \]

**Theorem 7.19** Let $G$ be a finite group. Then the Frattini subgroup $\Phi(G)$ is nilpotent.

**Proof**: Let $P$ be a Sylow $p$-subgroup of $\Phi(G)$. The Frattini Argument (Lemma 6.35) gives

$$G = N_G(P) \Phi(G).$$

If $N_G(P) \neq G$, then there is a maximal proper subgroup $M$ of $G$ with $N_G(P) \leq M < G$. By definition, $\Phi(G) \leq M$. Hence

$$N_G(P) \Phi(G) \leq M < G,$$

contrary to above. Therefore $N_G(P) = G$. Hence $P \trianglelefteq G$, and so in particular $P \trianglelefteq \Phi(G)$. Therefore $\Phi(G)$ is nilpotent by Theorem 7.13. \[ \Box \]

We have used one property of the Frattini subgroup twice now, so it is worth drawing attention to it.

**Definition 7.20** A subset $S$ of a group $G$ is a set of non-generators if it can always be removed from a set of generators for $G$ without affecting the property of generating $G$.  

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Thus $S$ is a set of non-generators if
\[ G = \langle X, S \rangle \quad \text{implies} \quad G = \langle X \rangle \]
for all subsets $X \subseteq G$.

**Lemma 7.21** The Frattini subgroup $\Phi(G)$ is a set of non-generators for a finite group $G$.

**Proof:** Let $G = \langle X, \Phi(G) \rangle$. If $\langle X \rangle \neq G$, then there exists a maximal subgroup $M$ of $G$ such that $\langle X \rangle \leq M < G$. By definition of the Frattini subgroup, $\Phi(G) \leq M$. Hence $X \cup \Phi(G) \subseteq M$, so $\langle X, \Phi(G) \rangle \leq M < G$ which contradicts the assumption. Therefore $G = \langle X \rangle$ and so we deduce $\Phi(G)$ is a set of non-generators for $G$. $\square$

**Theorem 7.22** Let $G$ be a finite group. Then $G$ is nilpotent if and only if $G/\Phi(G)$ is nilpotent.

**Proof:** By Lemma 7.5, a homomorphic image of a nilpotent group is nilpotent. Consequently if $G$ is nilpotent, then $G/\Phi(G)$ is nilpotent.

Conversely suppose $G/\Phi(G)$ is nilpotent. Let $P$ be a Sylow $p$-subgroup of $G$. Then $P\Phi(G)/\Phi(G)$ is a Sylow $p$-subgroup of $G/\Phi(G)$. Hence
\[ P\Phi(G)/\Phi(G) \trianglelefteq G/\Phi(G), \]
as $G/\Phi(G)$ is nilpotent. Therefore
\[ P\Phi(G) \trianglelefteq G \]
by the Correspondence Theorem. Now $P$ is a Sylow $p$-subgroup of $P\Phi(G)$ (as even $G$ has no larger $p$-subgroups), so we apply the Frattini Argument (Lemma 6.35) to give
\[ G = N_G(P) \cdot P\Phi(G). \]
Therefore
\[ G = N_G(P)\Phi(G) \]
(as $P \leq N_G(P)$). Now as $\Phi(G)$ is a set of non-generators for $G$ (see Lemma 7.21), we deduce
\[ G = N_G(P). \]
Thus $P \trianglelefteq G$. Hence $G$ is nilpotent by Theorem 7.13. $\square$