Primitive groups and maximal subgroups

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Main Goal of Finite Group Theory?

Classify all finite groups

Determine list L containing all finite groups, and for each isomorphism class of groups describe all ways it occurs in L.

. . . by order?

Small Groups Library [Besche, Eick & O'Brien]: complete for $|G| \leq 2000$, if $|G| \neq 1024$, plus various "nicely factorising" orders.

. . . by structure?

Classification of Finite Simple Groups describes all finite simple groups – but we don't know all ways of putting simples together.

. . . by permutation or matrix representation?

Given G, determine all embeddings of G in S_n or $\mathrm{GL}_d(q)$; given S_n or $\mathrm{GL}_d(q)$, describe all subgroups.

Mostly Definitions

The degree of $G \leq S_n$ is n (with $n < \infty$); write $\underline{n} = \{1, \ldots, n\}$.

If for all $\alpha, \beta \in \underline{n}$ there exists $g \in G$ with $\alpha g = \beta$ then G is transitive. Otherwise G is intransitive.

$$G = \langle (1,2,3,4), (2,4) \rangle \leq S_4 \quad \text{transitive}$$
$$H = \langle (1,2,3,4)(5,6), (2,4) \rangle \leq S_6 \quad \text{intransitive}$$

A point stabiliser of $G \leq S_n$ is

$$G_\alpha = \{g \in G \ : \ \alpha g = \alpha\},$$

for $\alpha \in \underline{n}$. All point stabilisers are subgroups of G.

$$G_1 = \langle (2,4) \rangle = H_1.$$

Let's classify all permutation groups!

n	2	3	4	5	6	7	8	9	10	11	12
$\#G$	2	4	11	19	56	96	296	554	1593	3094	10723

Theorem (Pyber '93)

The number of permutation groups of degree n is bounded below by $2^{(1/16+o(1))n^2}$.

Pyber conjectures: This bound is sharp

Lemma

An intransitive group is a subdirect product of transitive groups.

(subdirect product: subgp of direct prod that projects onto each factor)

. . . so we restrict ourselves to transitive groups.

Classifications of transitive groups

1858 prize question of the Académie des Sciences:

Quels peuvent être les nombres de valeurs des fonctions biens définies qui contiennent un nombre donné de lettres, et comment peut-on former les fonctions pour lesquelles il existe un nombre donné de valeurs?

That is: What are the indices of all subgroups of S_n?
Three submissions in 1860, no prize awarded!
Classifications of transitive groups of low degree:

$$n \leq 30, \text{ Hulpke 2005} \qquad n \leq 32, \text{ Holt 2008.}$$

Problems:

- Number of groups grows very fast: $n = 32$ gives $2,801,324$.
- Of these, 98% have order 2^i for some i – not much variety.

Primitivity and Imprimitivity

Let $\Delta \subset \underline{n}$ with $|\Delta| > 1$, and let $G \leq S_n$ be transitive.
If for all $g \in G$ either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \emptyset$ then:

- Δ is a block for G,
- $\{\Delta g \ : \ g \in G\}$ is a system of imprimitivity for G.

If G has no blocks then G is primitive; otherwise G is imprimitive.

1. $G = \langle (1, 2, 3, 4, 5, 6) \rangle \leq S_6$ is transitive.
   - $\{1, 4\}$ is a block for G, so G is imprimitive.
   - The system of imprimitivity induced by $\{1, 4\}$ is $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$.
   - $\{1, 3, 5\}$ also a block: systems of imprimitivity not unique.
2. $G = \langle (1, 2, 3, 4), (1, 2) \rangle = S_4$ is primitive.

Some remarks on point stabilisers

Lemma

Let $G \leq S_n$ be transitive.

- *The degree, n, of G is equal to $|G|/|G_\alpha|$, for any $\alpha \in \underline{n}$.*
- *For all $\alpha, \beta \in \Omega$, G_α and G_β are conjugate subgroups of G.*
- *G is primitive if and only if G_α is a maximal subgroup of G.*

(conjugate: there is a $g \in G$ such that $g^{-1}G_\alpha g = G_\beta$)
($H \leq G$ is maximal if the only proper subgroup of G containing H is H)

G transitive $\Rightarrow$ can identify elements of $\underline{n}$ with cosets of G_1 in G:

$$1 \leftrightarrow G_1,\ 2 \leftrightarrow G_1 g_2 = \{gg_2 \ : \ g \in G_1\},\ 3 \leftrightarrow G_1 g_3, \ldots$$

where $1g_i = i$.

…so studying permutation reps $\leftrightarrow$ studying subgroups.

Permutation Group Reductions

Theorem

Let G be imprimitive, Δ a block, $\Gamma = \{\Delta g \ : \ g \in G\}$.

- *Let $G_{\{\Delta\}}$ be the setwise stabiliser of Δ.*
- *Let G^{Γ} be the permutation group induced by G on Γ.*

Then G is a subgroup of $G_{\{\Delta\}} \wr G^{\Gamma}$.

- Consider $C_6 = \langle (1,2,3,4,5,6) \rangle$
  1. Has blocks $\{\{1,4\},\{2,5\},\{3,6\}\}$, so $C_6 \leq C_2 \wr C_3$.
  2. Has blocks $\{\{1,3,5\},\{2,4,6\}\}$, so $C_6 \leq C_3 \wr C_2$.

Implication: Primitive groups are the building blocks of all permutation groups.

What are the primitive groups of degree n?

What are the primitive groups of degree n?

- 1871. Jordan, $n \leq 17$.
  Some omissions in degrees 9, 12, 15, 16, 17.
- 1874. Jordan, $n = 19$.
  Stated that are alternating, symmetric or of affine type.
- 1893. Cole, $n = 9$ corrected.
- 1895–1900. Miller, $n = 12, \ldots, 17$ corrected.
- By 1912. $n \leq 20$.
- Hiatus
- 1960s. Start of 'practical' computational algebra.
- 1970. Sims, rechecked $n \leq 20$. Computational.
- By 1970. Sims, $n \leq 50$. Turned into databases in CAYLEY, GAP and MAGMA.

The Classification and Beyond

1980s: Classification of Finite Simple Groups; the O'Nan–Scott Theorem.

Problem now splits into two main parts:

1. Classify the maximal subgroups of the almost simple groups of index n (insoluble socles).
2. Classify the irreducible subgroups of $\mathrm{GL}_d(p)$, where $p^d = n$ (soluble socles).

- 1988. Dixon & Mortimer. Insoluble socles for $n < 1000$.
- 1991. Short. Primitive soluble groups for $n < 256$.
- 2003. Eick & Höfling. Primitive soluble groups for $n < 6561$.
- 2003. Roney-Dougal & Unger. Soluble socles for $n < 1000$.
- 2005. Roney-Dougal. All for $n < 2500$.
- 2009. Coutts, Quick & Roney-Dougal. All for $n < 4096$.

The groups with soluble socles

(socle: subgroup of G generated by the minimal normal subgroups of G.)

Theorem

Let $G \leq S_n$ be primitive, with soluble socle. Then $n = p^d$ and

- $G \cong \mathbb{F}_p^d : H$ *for some* $H \leq \mathrm{GL}_d(p)$*;*
- H *acts irreducibly on* $\mathbb{F}_p^d$*, with* $G_\alpha \cong H$*.*

(irreducible: stabilises no proper nonzero subspace of $\mathbb{F}_p^d$.)
So classifying such $G \leftrightarrow$ classifying irreducible subgrps of $\mathrm{GL}_d(p)$.

- Largely computational problem.
- Hardest task is determining when groups are conjugate.
- Large subproblem: H soluble, so G is soluble.
  - Then can use techniques for polycyclic groups.
  - Task complete for $p^d < 6561$ [Eick & Höfling].

The classification of finite simple groups

Theorem (Classification of finite simple groups)

Let G be a finite simple group. Then one of the following holds:

- *G is cyclic of prime order;*
- *G is an alternating group A_n for $n \geq 5$;*
- *G is group of Lie type, and either:*
  - *G is a classical group, one of $\mathrm{PSL}_d(q)$, $\mathrm{PSp}_d(q)$, $\mathrm{PSU}_d(q)$ or $\mathrm{P}\Omega^{\epsilon}_d(q)$, for $\epsilon \in \{+, -, \circ\}$ – these are parametrised by dimension and field size;*
  - *G is an exceptional group: there are ten families of such groups, each parametrised by field size;*
- *G is one of 26 sporadic groups.*

G is almost simple if there exists a nonabelian simple group S with $S \trianglelefteq G \leq \mathrm{Aut}(S)$.

Classifying Maximal Subgroups

Investigating the structure of the almost simple groups is one of the main jobs arising from CFSG.

The maximal subgroups of the almost simple groups determine the primitive groups with insoluble socles.

Lemma

Each maximal subgroup of A_n *and* S_n *is intransitive, imprimitive, or primitive.*

- Intransitive and imprimitive maximals – easy to both characterise and construct.
- Primitive groups are divided into 5 classes by the O'Nan–Scott Theorem.
- Four classes come with explicit largest groups, the fifth consists of almost simple groups.

Alternating and symmetric groups

Example (Maximal subgroups of A_8)

In A_8	*What is it?*	*In* S_8
A_7	1-*point stabiliser*	S_7
S_6	2-*point stabiliser*	$S_2 \times S_6$
$(3 \times A_5) : 2$	3-*point stabiliser*	$S_3 \times S_5$
$(S_4 \wr S_2) \cap A_8$	*block size 4*	$S_4 \wr S_2$
$\mathrm{AGL}_3(2)$ *(2 copies)*	*primitive, affine*	—
—	*primitive, almost simple*	$\mathrm{PGL}_2(7)$
—	*block size 2*	$S_2 \wr S_4$

Liebeck, Praeger & Saxl determine maximals of A_d and S_d, given the almost simple primitive grps of degree d.

- So we're fine up to $d = 4095$ – which means $n \gg 4095$.

Sporadic groups

Standard generators for G are a generating sequence X for G that is specified up to automorphisms of G.

- Standard gens of M_{22} are a, b where $|a| = 2$, b is in class 4A, $|ab| = 11$ and $|ababb| = 11$.

Theorem (Wilson *et al.*)

T – almost simple with sporadic socle. Then we know:

- *algorithms to find standard generators of T;*
- *the maximal subgroups of T (if $T \neq \mathrm{M}$);*
- *words in standard gens for the gens of each maximal subgroup (if $\mathrm{Soc}(T) \notin \{\mathrm{Co}_1, \mathrm{HN}, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}', B, M\}$).*

All maximal subs of the Monster of order > 5515776 are known, so finding primitive representations on $< 10^{47}$ points is easy.

Exceptional groups

Recall: there are 10 infinite families of exceptional simple groups.

Each family is parametrised by field size & has a "natural" dimension.

Families of maximal subgroups reasonably well understood.

Maximal subgroups known explicitly for some low-dimension families: e.g. $\mathrm{Sz}(q)$, $G_2(q)$, ${}^2F_4(q)$.

Still lots of work to be done. However, the exceptional groups do not have many representations as permutation groups on a small number of points.

Classical groups

Recall: there are four doubly-infinite families of classical groups.

Each parametrised by dimension d, field size q, (and type).

Aschbacher's theorem divides all subgroups of a classical group G, with a few exceptions, into nine classes.

- Groups in the first eight classes are geometric.
- The final class, $\mathcal{S}$, consists (roughly) of absolutely irreducible groups that are almost simple modulo scalars.

Geometric subgroups:

- lots known about structure;
- for $d \geq 13$, know when are maximal [Kleidman & Liebeck].

Quasisimple $\mathcal{S}$ groups known for $d \leq 250$. [Hiss & Malle, Lübeck].
(Quasisimple $=$ simple modulo scalars, and perfect.)

Conjugacy of maximal subgroups

Aschbacher's theorem BΔ states (roughly) that for a classical group C, in $\mathrm{N}_{\mathsf{GL}_d(q)}(C)$ there is a unique conjugacy class of each candidate geometric maximal, with the exception of some known subgroups of $\mathsf{GL}_d(q)$, $\mathrm{PSp}_4(2^i)$ and $\mathrm{P}\Omega_8^+(q)$.

(Subgroups H and K are conjugate in G if $\exists g \in G$ with $g^{-1}Hg = K$.)

- The conjugacy classes of candidate geometric maximal subgroups of the almost simple classical groups are understood [Kleidman & Liebeck].
- The conjugacy of groups in $\mathcal{S}$ has to be determined on a case-by-case basis.

Constructing the maximals

Theorem (Holt & Roney-Dougal 2005, 2009)

Let $G \leq \mathrm{GL}_d(q)$ be classical, with $\Omega \trianglelefteq G$ the corresponding quasisimple classical. Then the intersections with Ω of the geometric maximal subgroups of G, up to conjugacy in $\mathrm{N}_{\mathrm{GL}_d(q)}(\Omega)$, can be constructed in $O(d^3 \log d \log^2 q)$ field operations.

The groups in $\mathcal{S}$ fall into two classes:

- individual groups that lie in a classical group of a fixed dimension (group and d fixed; q may vary)
- families of simple groups that lie in other families of classical group (the group itself varies, q varies; d fixed).

Nickerson & Holt have been making representations of groups in the first class, with the field size as an input parameter. Groups in the second class can usually be made as needed.

Which maximals to construct?

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Which maximals to construct?

Current Project: Classify the maximal subgroups of the almost simple classical groups for $d \leq 12$, all q: with Bray & Holt.

- $d \leq 5$ – many results from ~ 1930, but lots of errors.
- Unpublished work of Kleidman – contains results for the simple groups, but again many errors, and no proofs.

Plan is to redo from scratch – check against old results but not rely on them.

- Geometrics: done linear, symplectic, unitary groups.
- $\mathcal{S}$ groups: largely classified maximals, many proofs still to write!
- Constructing all maximal subgroups in MAGMA as we go.

London Mathematical Society Lecture Note Series, *to appear*.