PARTIAL SOLUTIONS FOR ‘FUNDAMENTALS OF PURE MATHEMATICS’ SEPTEMBER 2007 RESIT EXAMINATION

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1. (a) (a) Dense: if \( x, y \in \mathbb{Q}^+ \) with \( x < y \) then \( x < (x + y)/2 < y \) and \( (x + y)/2 \in \mathbb{Q}^+ \). (b) Not dense: \( 1, 2 \in \mathbb{N} \) but there is no \( x \in \mathbb{N} \) with \( 1 < x < 2 \). (c) Dense: similar to (a), observe that \( 3 \leq (x + y)/2 \leq 4 \). (d) Not dense \( 2, 3 \) lie in the set but there is no member of the set with \( 2 < x < 3 \).

(b) It lies in \( B \) since \( a = 5a = a + 4a < a + 4b < \frac{b + 4b}{5} = \frac{5b}{5} = b \).

(c) \( A \) is dense. Pick \( \frac{p}{5^m}, \frac{q}{5^n} \in A \) with \( \frac{p}{5^m} < \frac{q}{5^n} \). Then
\[
\frac{p/5^m + 4q/5^n}{5} = \frac{5^mp + 4 \cdot 5^m q}{5^m+n+1} \in A,
\]
and by the previous part,
\[
\frac{p}{5^m} < \frac{(p/5^m + 4q/5^n)}{5} < \frac{q}{5^n}.
\]

(d) Suppose \( r \in \mathbb{Q}^+, r + \frac{1}{r} \in \mathbb{Z} \). Let \( r = p/q \) with \( p \) and \( q \) coprime (no common factor except 1) and \( k \in \mathbb{Z} \) is such that
\[
k = r + \frac{1}{r} = \frac{p}{q} + \frac{q}{p}.
\]
So \( kpq = p^2 + q^2 \). Therefore \( p^2 = q(kp - q) \) and \( q^2 = p(kq - p) \). Suppose \( t \) is a prime factor of \( q \). Then it is a prime factor of \( p^2 \) and thus (by elementary facts about primes) a factor of \( p \). This contradicts \( p \) and \( q \) being prime. So \( q \) has no prime factors, i.e. \( q = 1 \).

Similar reasoning shows \( p = 1 \). So \( r = p/q = 1 \).

(e) Let \( k \in \mathbb{N} \) and suppose \( r + 1/r = k \). Then \( r^2 - kr + 1 = 0 \). This equation has two solutions \( r_k, r'_k \). List the solutions for all \( k \):
\[
r_1, r'_1, r_2, r'_2, r_3, r'_3, \ldots
\]
Thus the set of all positive reals \( r \) with \( r + 1/r \) being an integer is countable.

2. (a) A nonempty subset \( A \) of \( \mathbb{Q} \) is a Dedekind cut if it (a) is bounded above, (b) has no maximum, and (c) is closed downwards.

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(b) Many examples, for instance \( G = \{ x \in \mathbb{Q} : x < 0 \} \), \( H = \{ 1 \} \).

(c) The relation \( \leq \) inherits reflexivity, anti-symmetry, and transitivity from the order \( \subseteq \). Let \( A, B \) be cuts. We want to show that either \( A \leq B \) or \( B \leq A \). So assume \( A \nleq B \). Then \( A \nsubseteq B \). So there exists \( a \in A \setminus B \). Let \( b \in B \). Then \( b < a \) (since \( b > a \) would imply \( a \in B \) since \( B \) is closed downwards). So \( b \in A \) since \( A \) is closed downwards. So \( b \in B \implies b \in A \). Hence \( B \subseteq A \), i.e. \( B \leq A \). So \( \leq \) is a total order.

(d) The order is not total. For instance, the sets \( \{ 0 \} \) and \( \{ 1 \} \) are not comparable.

(e) \( \mathfrak{T} \) is bounded above by \( r \). It has no maximum, since its least upper bound \( r \) does not lie in \( \mathfrak{T} \). It is clearly closed downwards. So \( \mathfrak{T} \) is a cut.

(f) Observe:

\[
A = \{ x \in \mathbb{Q} : x^2 < 2 \} \cup \{ x \in \mathbb{Q} : x < 0 \} = \{ x \in \mathbb{Q} : x \geq 0, x^2 < 2 \} \cup \{ x \in \mathbb{Q} : x < 0 \}
\]

Now suppose \( A = \mathfrak{T} \). Then

\[
A^2 = (\mathfrak{T})^2 = \{ x \in \mathbb{Q} : 0 < x < 2 \} \cup \{ x \in \mathbb{Q} : x < 0 \} = \{ x \in \mathbb{Q} : x > r^2 \} = \{ x \in \mathbb{Q} : x < 2 \} = \{ x \in \mathbb{Q} : x < r^2 \}.
\]

So \( r^2 = 2 \), which is a contradiction since \( r \) is rational.

3. (a) Proceed as follows:

\[
0.1353353\ldots = 0 + \frac{1}{10} + \frac{3}{10^2} + \frac{5}{10^3} + \frac{3}{10^4} + \frac{5}{10^5} + \ldots
\]

\[
= \frac{1}{10} + \frac{35}{10^3} + \frac{35}{10^3} + \frac{35}{10^6} + \ldots
\]

\[
= \frac{1}{10} + \frac{35}{10^3} \left[ 1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots \right]
\]

\[
= \frac{1}{10} + \frac{35}{10^3} \left[ \frac{1}{1 - \frac{1}{10^2}} \right]
\]

\[
= \frac{1}{10} + \frac{35}{10^3} \cdot \frac{100}{99}
\]

\[
= \frac{1}{10} + \frac{35}{990}
\]

\[
= \frac{134}{990}.
\]

(b) \( r \) is rational. The second-last digits of the numbers 10, 11, 12, \ldots form a periodic sequence: ten digits 1, ten digits 2, ten digits 3, \ldots ten digits 9, and repeat. The numbers with periodic decimal expansions are precisely the rational numbers.
(c) \( s \) is not rational. The first digits of the natural numbers do not form a periodic sequence. For any \( k \), the sequence of first digits of the natural numbers starting from \( 10^k \) begins with \( 10^k \) digits \( 1 \). So we can always find a string of 1s longer than any supposed period.

(d) There are many different ways to answer this question. For any \( n \in \mathbb{N} \), \( A \) contains

\[
x_n = 0.1 \ldots 15 \underbrace{1 \ldots 1}_{n} \ldots 15 \underbrace{1 \ldots 1}_{n+1} \ldots 15 \ldots \n+2
\]

The numbers \( x_n \) are all rational (they have periodic decimal expansion) and are all distinct. So \( A \) contains infinitely many rational numbers.

For any \( n \in \mathbb{N} \), \( A \) contains

\[
y_n = 0.1 \ldots 15 \underbrace{1 \ldots 1}_{n} \ldots 15 \underbrace{1 \ldots 1}_{n+1} \ldots 15 \ldots \n+2
\]

The numbers \( y_n \) are all irrational (they do not have periodic decimal expansion) and are all distinct. So \( A \) contains infinitely many irrational numbers.

(e) \( A \) has a minimum, namely \( 0.11111 \ldots \), and a maximum, namely \( 0.55555 \ldots \).

(f) \( A \) is not dense. For example \( 0.15555 \ldots \) and \( 0.51111 \ldots \) are both in \( A \), but no real number between these two values lies in \( A \).

(g) \( A \) is uncountable.

4. (a) An infinite set \( X \) is countable if it has the same cardinality as \( \mathbb{N} \): that is, if there is a bijection from \( X \) to \( \mathbb{N} \). An infinite set \( X \) is uncountable if it has greater cardinality that \( \mathbb{N} \): if there is no bijection from \( \mathbb{N} \) to \( X \).

(b) Many examples exist, for instance \( S = \mathbb{N} \), \( T = 2\mathbb{N} = \{2n : n \in \mathbb{N}\} \), \( S \setminus T = \{2n - 1 : n \in \mathbb{N}\} \).

(c) For any set \( X \), \(|X| < |P(X)|\).

(d) Define a map \( f : B \rightarrow P(\mathbb{N}) \) by

\[
x_1, x_2, x_3, \ldots \mapsto \{n \in \mathbb{N} | x_n = 1\}.
\]

The mapping \( f \) is a bijection: its inverse is \( g : P(\mathbb{N}) \rightarrow B \), where

\[
Y \mapsto y_1, y_2, y_3, \ldots \text{ where } \begin{cases} y_i = 1 & \text{if } i \in Y \\ y_i = 0 & \text{if } i \notin Y. \end{cases}
\]

(Alternatively, prove that \( f \) is a bijection by showing it is injective and surjective.) So \(|P(\mathbb{N})| = |B|\).

(e) \(|B| = |P(\mathbb{N})| > |\mathbb{N}| \) by Cantor’s Theorem and the previous part. So \(|B| > |\mathbb{N}| \), i.e. \( B \) is uncountable.