Fundamentals of Pure Mathematics

These slides can be downloaded from:

www-groups.mcs.st-andrews.ac.uk/~alanc/teaching/default.html
Theorem 24.1

\[ |(0, 1)| = |\mathcal{P}(\mathbb{N})| \cdot (So \ \mathcal{P}(\mathbb{N}) \text{ is uncountable.}) \]

Proof.

Define \( f : (0, 1) \rightarrow \mathcal{P}(\mathbb{N}) \) by

\[
0.a_1 a_2 a_3 \ldots \mapsto \{ i : a_i = 1 \},
\]

where \( a_j \in \{0, 1\} \).

\( f \) is an injection, so \( |(0, 1)| \leq |\mathcal{P}(\mathbb{N})| \).
Proof (cont.)

Define \( g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1) \) as follows,

\[
g(X) = 0.a_10a_20a_3\ldots \quad (i \in X \implies a_i = 1, i \notin X \implies a_i = 0).
\]

\( g \) is an injection.

So, by the Schröder–Bernstein theorem, \(|(0, 1)| = |\mathcal{P}(\mathbb{N})|\).

Why not \( g(X) = 0.a_1a_2a_3\ldots \)? Because then

\[
g(\{1\}) = 0.1000\ldots = 0.0111\ldots = g(\mathbb{N} - \{1\}).
\]

(cf. 0.1000\ldots = 0.0999\ldots for decimal expansions.)
Proposition 24.2

Let $P$ be the set of all subsets of $\mathbb{N}$ of size 2. Then $P$ is countable.

Proof.

Clearly $|P| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Since $\mathbb{N}$ has the smallest cardinality of any infinite set, $|P| = |\mathbb{N}|$. 
Proposition 24.3

Let $P_n$ be the set of all subsets of $\mathbb{N}$ of size $n$. For any $n \in \mathbb{N}$, $P_n$ is countable.

Proof.

Clearly $|P_n| \leq |P_{n-1} \times \mathbb{N}|$. By induction, $P_{n-1}$ is countable. So $|P_n| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. So $|P| = |\mathbb{N}|$. 

\qed
Recall that $\mathcal{P}X$, the power set of $X$, is defined by

$$\mathcal{P}X = \{ Y : Y \subseteq X \}.$$ 

The *finitary power set* of $X$, denoted $\mathcal{P}_F X$, is the set whose members are finite subsets of $X$

$$\mathcal{P}_F X = \{ Y : Y \subseteq X, \ Y \text{ is finite} \}$$

Notice that

$$\mathcal{P}_F \mathbb{N} = \{ \emptyset \} \cup \bigcup_{n \in \mathbb{N}} \mathcal{P}_n.$$
Proposition 24.4

\( \mathcal{P}_F(\mathbb{N}) \) is countable.
More cardinality and countability

Countable unions of countable sets

**Theorem 24.5**

If $S_i$ is countable for each $i \in \mathbb{N}$, then

$$S = \bigcup_{i \in \mathbb{N}} S_i$$

is countable.

**Proof.**

Let $f_i : S_i \rightarrow \mathbb{N}$ be a bijection. Define $f : S \rightarrow \mathbb{N} \times \mathbb{N}$ by

$$x \mapsto (i, f_i(x))$$

where $x \in S_i$ but not in $S_1 \cup \cdots \cup S_{i-1}$. This map is an injection, so $|S| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. So $|S| = |\mathbb{N}|$. 

\hfill \square
Proof of Proposition 24.7.

$\mathcal{P}_F(\mathbb{N})$ is the union of the various countable sets $P_n$ and the 1-element set $\{\emptyset\}$, so is itself countable.
This is the end of the course, but we have a few matters to clear up:

- Surprise Egg Competition winners
- Feedback survey
- Next week — revision lectures on Monday and Tuesday (Jan 2007 paper)
- Next week — tutorial & examples class