Definition 18.1

\[ e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}. \]
Irrationality of $e$

Theorem 18.2

$e$ is irrational.

Proof.

Suppose $e = a/b$ where $a, b \in \mathbb{Z}$. Let

$$m = b! \left( e - \sum_{n=0}^{b} \frac{1}{n!} \right)$$

$$= b! \left( \frac{a}{b} - \left( \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{b!} \right) \right).$$

Then $m \in \mathbb{Z}$. 
Irrationality of $e$

Proof (continued).

$$0 < m = b! \left( e - \sum_{n=0}^{b} \frac{1}{n!} \right) = b! \left( \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!} \right)$$

$$= b! \left( \frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \frac{1}{(b+3)!} + \cdots \right)$$

$$= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots$$

$$< \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots$$

$$= \frac{1}{b+1} \cdot \frac{1}{1 - \frac{1}{b+1}} = \frac{1}{b} < 1.$$ 

So $0 < m < 1$, which contradicts $m \in \mathbb{Z}$. □
Definition 18.3

$\pi$ is the smallest positive root of $\sin(x)$.

Strictly speaking, more work is needed...

- Check that $\sin$ is continuous.
- Check that it has positive roots.

Recall that a circle of radius 1 has

- Area $\pi$.
- Circumference $2\pi$
Irrationality of $\pi$

Theorem 18.4

$\pi^2$, and hence $\pi$ itself, is irrational.
Irrationality of $\pi$

Define a sequence of polynomials

$$p_n(x) = \frac{x^n(1 - x)^n}{n!} \quad (n = 1, 2, 3, \ldots).$$

Because of the factor $x^n$, $p_n(x)$ has no terms of degrees $0, 1, \ldots, n - 1$. Hence

$$p_n(x) = \frac{1}{n!} \sum_{k=n}^{2n} a_k x^k,$$

where $a_k \in \mathbb{Z}$.

**Lemma 18.5**

For $0 < x < 1$ and $n \geq 1$, $0 < p_n(x) < \frac{1}{n!}$. 
Irrationality of $\pi$

Recall that $p_n^{(h)}$ is the $h$-th derivative of $p_n$. Now,

$$p_n(x) = \frac{1}{n!} \sum_{k=n}^{2n} a_k x^k,$$

so

$$p_n^{(k)}(0) = 0$$

for $0 \leq k < n$ and for $k > 2n$, while for $n \leq k \leq 2n$ we have

$$p_n^{(k)}(0) = \frac{k!}{n!} a_k \in \mathbb{Z}.$$

**Lemma 18.6**

For all $n \geq 1$ and $k \geq 0$, $p_n^{(k)}(0) \in \mathbb{Z}$.
Irrationality of $\pi$

Note that

$$p_n(x) = \frac{x^n(1-x)^n}{n!} = \frac{(1-x)^n x^n}{n!} = \frac{(1-x)^n(1-(1-x))^n}{n!} = p_n(1-x).$$

It follows (by the chain rule for derivatives) that

$$p_n^{(k)}(x) = (-1)^k p_n^{(k)}(1-x) \quad (k = 1, 2, 3 \ldots).$$

Hence

$$p_n^{(k)}(1) = (-1)^k p_n^{(k)}(0),$$

which is an integer by the previous lemma.

Lemma 18.7

For all $n \geq 1$ and all $k \geq 0$, the number $p_n^{(k)}(1)$ is an integer.
Irrationality of $\pi$ 

Suppose $\pi^2 = a/b$ with $a, b \in \mathbb{Z}$.

Define

$$G(x) = b^n \left( \pi^{2n} p_n(x) - \pi^{2n-2} p_n^{(2)}(x) + \ldots + (-1)^n p_n^{(2n)}(x) \right).$$

Hence

$$G''(x) = b^n \left( \pi^{2n} p_n^{(2)}(x) - \pi^{2n-2} p_n^{(4)}(x) + \ldots \right.$$

$$\left. \ldots + (-1)^{n-1} \pi^2 p_n^{(2n)}(x) + (-1)^n p_n^{(2n+2)}(x) \right).$$

Note that the last term in the above sum is in fact equal to 0.
Irrationality of $\pi$

\[ G(x) = b^n \left( \pi^{2n} p_n(x) - \pi^{2n-2} p_n^{(2)}(x) + \ldots + (-1)^n p_n^{(2n)}(x) \right) \]

\[ G''(x) = b^n \left( \pi^{2n} p_n^{(2)}(x) - \pi^{2n-2} p_n^{(4)}(x) + \ldots + (-1)^{n-1} \pi^2 p_n^{(2n)}(x) \right) \]

\[ \frac{d}{dx} \left( G' \sin \pi x - \pi G(x) \cos \pi x \right) \]

\[ = G''(x) \sin \pi x + \pi G'(x) \cos \pi x - \pi G'(x) \cos \pi x + \pi^2 G(x) \sin \pi x \]

\[ = \left( G''(x) + \pi^2 G(x) \right) \sin \pi x \]

\[ = b^n \pi^{2n+2} p_n(x) \sin \pi x \]

\[ = \pi^2 a^n p_n(x) \sin \pi x. \]
Irrationality of $\pi$

\[ \frac{d}{dx} \left( G' \sin \pi x - \pi G(x) \cos \pi x \right) = \pi^2 a^n p_n(x) \sin \pi x. \]

Therefore:

\[ \pi \int_0^1 a^n p_n(x) \sin(\pi x) \, dx \]
\[ = \left. \left( \frac{G'(x) \sin \pi x}{\pi} - G(x) \cos \pi x \right) \right|_0^1 \]
\[ = G(0) + G(1), \]

which is an integer.
Irrationality of $\pi$

On the other hand, by choosing $n$ large enough to ensure

$$\pi \frac{a^n}{n!} < 1$$

one obtains

$$0 < \pi \int_0^1 a^n p_n(x) \sin(\pi x) \, dx < \pi \int_0^1 \frac{a^n}{n!} \, dx = \pi \frac{a^n}{n!} < 1,$$

which is a contradiction.