Theorem 17.1

Every non-constant complex polynomial

\[ p(x) = a_n x^n + \ldots + a_1 x + a_0 \ (a_i \in \mathbb{C}, \ n \geq 1, \ a_n \neq 0) \]

has a zero in \( \mathbb{C} \), i.e. there exists \( z \in \mathbb{C} \) such that \( p(z) = 0 \).
Facts 17.2

- Every complex polynomial is a continuous function \( C \rightarrow C \).
- The modulus of a complex polynomial \( p(x) \) attains its greatest lower bound, i.e. there exists \( c \in C \) such that
  \[
  |p(c)| = \inf\{|p(z)| : z \in C\}. \text{ [This is not true of all functions: e.g. } \]
  \[
  f(x) = \frac{1}{|x|} \text{ has g.l.b. 0 but doesn’t take the value 0.} \]
Lemma 17.3

Let \( g(x) \) be a complex polynomial with \( g(0) = 0 \), let \( k \in \mathbb{N} \), let \( b \in \mathbb{C} \), \( b \neq 0 \), and let

\[
h(x) = 1 + bx^k + x^k g(x).
\]

Then there exists \( u \in \mathbb{C} \) such that \( |h(u)| < 1 \).
Proof.

By Theorem 16.3 (Existence of Roots) there exists \( d \in \mathbb{C} \) such that

\[ d^k = -\frac{1}{b}. \]

Then for all \( t \in \mathbb{R} \) with \( 0 < t \leq 1 \) we have

\[
| h(dt) | = | 1 + b d^k t^k + d^k t^k g(dt) | \\
= | 1 - t^k + d^k t^k g(dt) | \\
\leq | 1 - t^k | + | d^k t^k g(dt) | \\
= 1 - t^k + t^k | d^k g(dt) |. 
\]

(since \( d^k = -1/b \))

(by Theorem 15.7(i))

(by Theorem 15.7(ii))
Proof (continued).

\[ |h(dt)| \leq 1 - t^k + t^k |d^k g(dt)| \]

\( g \) is continuous and \( g(0) = 0 \), so \( g(dt) \) can be made arbitrarily small by choosing small \( t \). i.e. there exists \( 0 < \delta < 1 \) such that for all \( t \) satisfying \( 0 < t < \delta \) we have

\[ |d^k g(dt)| \leq \frac{1}{2}. \]

So for any such \( t \),

\[ |h(dt)| \leq 1 - t^k + \frac{1}{2} t^k = 1 - \frac{1}{2} t^k < 1, \]

proving the lemma.
Lemma 17.4

Let $f(x)$ be a non-constant complex polynomial. Then for every $c \in \mathbb{C}$ such that $f(c) \neq 0$ there exists another $c' \in \mathbb{C}$ such that $|f(c')| < |f(c)|$. 
Proof.

Define a new polynomial

\[ p(x) = \frac{f(c + x)}{f(c)} = b_nx^n + \ldots + b_1x + b_0. \]

Note that \( b_0 = 1 \). Let \( b_k \) be the first non-zero coefficient after \( b_0 \):

\[ p(x) = b_nx^n + \ldots + b_kx^k + 1 = 1 + b_kx^k + x^k g(x). \]

Clearly, \( g(x) \) satisfies \( g(0) = 0 \). So, by the previous Lemma, there exists \( u \in \mathbb{C} \) such that \( |p(u)| < 1 \). But then, for \( c' = c + u \), we have

\[ |f(c')| = |p(u)f(c)| = |p(u)||f(c)| < |f(c)|, \]

proving the lemma.
The Fundamental Theorem of Algebra

Proof of the Fundamental Theorem of Algebra

Theorem

Every non-constant complex polynomial

\[ p(x) = a_n x^n + \ldots + a_1 x + a_0 \quad (a_i \in \mathbb{C}, \ n \geq 1, \ a_n \neq 0) \]

has a zero in \( \mathbb{C} \), i.e. there exists \( z \in \mathbb{C} \) such that \( p(z) = 0 \).

Proof.

There exists \( c \in \mathbb{C} \) such that \( |p(c)| = \inf\{|p(z)| : z \in \mathbb{C}\} \).

Suppose that \( p(c) \neq 0 \).

Then there exists \( c' \in \mathbb{C} \) such that \( |p(c')| < |p(c)| \), a contradiction.