HONOURS B.Sc., M.A. AND M.MATH. EXAMINATION
MATHEMATICS AND STATISTICS
Paper MT3600 Fundamentals of Pure Mathematics
September 2007
Time allowed : Two hours
Attempt all FOUR questions
1. Let $X$ be a set, and let $\leq$ be a total order on $X$. We say that $X$ is dense if for any two $x, y \in X$ with $x < y$ there exists $z \in X$ such that $x < z < y$. In what follows $\leq$ always denotes the usual ordering on rational numbers.

(i) For each of the following sets determine whether it is dense or not: (a) $\mathbb{Q}^+$ (positive rationals); (b) $\mathbb{N}$ (positive integers); (c) $\{x \in \mathbb{Q} : 3 \leq x \leq 4\}$; (d) $\{x \in \mathbb{Q} : 1 \leq x \leq 2\} \cup \{x \in \mathbb{Q} : 3 \leq x \leq 4\}$. Justify your assertions. [4]

(ii) Let $a$ and $b$ be rational numbers satisfying $a < b$. To which of the following three sets does the number $\frac{a + 4b}{5}$ belong: $A = \{x \in \mathbb{Q} : x < a\}$, $B = \{x \in \mathbb{Q} : a < x < b\}$ or $C = \{x \in \mathbb{Q} : b < x\}$? Justify your answer. [1]

(iii) Consider the set
\[
A = \left\{ \frac{a}{5^n} : a, n \in \mathbb{Z}, \ n \geq 0 \right\}.
\]
Is $A$ dense? Justify your answer. [2]

(iv) Prove that $r = 1$ is the only positive rational number such that $r + \frac{1}{r}$ is an integer. [2]

(v) How many positive real numbers $x$ are there such that $x + \frac{1}{x}$ is an integer? Your answer should be one of: 0, 1, 2, 3, ..., countably infinite, or uncountably infinite, and you should justify it. [3]

2. (i) Define what it means for a set $A \subseteq \mathbb{Q}$ to be a Dedekind cut. [2]

(ii) Give an example of a set $G \subseteq \mathbb{Q}$ that is a Dedekind cut, and an example of a set $H \subseteq \mathbb{Q}$ that is not a Dedekind cut. [2]

(iii) On the set $\mathbb{R}$ of all Dedekind cuts, define a relation $\leq$ as follows:
\[
A \leq B \iff A \subseteq B.
\]
Prove that $\leq$ is an order and that this order is total. [2]

(iv) Let $\mathcal{P}(\mathbb{Q})$ be the set of all subsets of $\mathbb{Q}$. Is it true that $\subseteq$ is a total order on $\mathcal{P}(\mathbb{Q})$? Justify your answer. [1]

(v) For a rational number $r$ define the set
\[
\tau = \{x \in \mathbb{Q} : x < r\}.
\]
Prove that $\tau$ is a Dedekind cut. [2]

(vi) We proved in lectures that the set
\[
A = \{x \in \mathbb{Q} : x^2 < 2\} \cup \{x \in \mathbb{Q} : x < 0\}
\]
is a Dedekind cut. Prove that this Dedekind cut is not equal to any cut of the form \( r, r \in \mathbb{Q} \). [4]

3. (i) Express the number with the periodic decimal expansion 0.1353535... in the form \( \frac{m}{n}, m, n \in \mathbb{Z} \). [2]

(ii) Let \( r \) be the number 0.a1a2a3..., where \( a_n \) is the penultimate digit of \( n + 10 \) for every \( n \in \mathbb{N} \). Is \( r \) rational or irrational? Justify your answer. [2]

(iii) Let \( s \) be the number 0.a1a2a3..., where \( a_n \) is the first digit of \( n \) for every \( n \in \mathbb{N} \). Is \( s \) rational or irrational? Justify your answer. [2]

Let \( A \subseteq \mathbb{R} \) be the set of all real numbers with (infinite) decimal expansion of the form 0.d1d2d3... where \( d_i \in \{1, 5\} \) for all \( i \in \mathbb{N} \).

(iv) Prove that \( A \) contains infinitely many rational numbers, and also infinitely many irrational numbers. [2]

(v) Does \( A \) have a minimum (smallest element)? If so, what is it? Does \( A \) have a maximum (largest element)? [2]

(vi) Is \( A \) dense? Justify your answer. (For a reminder of the definition of density see Question 1.) [2]

(vii) Is the set \( A \) countable or uncountable? [1]

4. (i) Explain what is meant by saying that an infinite set is (a) countable; (b) uncountable. [2]

(ii) Give an example of a set \( S \) with a subset \( T \) such that \( S, T, \) and \( S \setminus T \) are all countable. [2]

(iii) State (without proof) Cantor’s Theorem. [2]

(iv) An infinite binary sequence is a sequence \( x_1, x_2, x_3, \ldots \), where \( x_n \in \{0, 1\} \) for all \( n \in \mathbb{N} \). Let \( B \) be the set of all infinite binary sequences. Prove that \( B \) has the same cardinality as \( \mathcal{P}(\mathbb{N}) \), the power set of the set of the natural numbers. [Hint: Suppose \( x_1, x_2, x_3, \ldots \) is a binary sequence. Consider the set \( \{n \in \mathbb{N} : x_n = 1\} \).] [4]

(v) Deduce that the set \( B \) is uncountable. [2]