HONOURS B.Sc., M.A. AND M.MATH. EXAMINATION
MATHEMATICS AND STATISTICS
Paper MT3600 Fundamentals of Pure Mathematics
January 2007
Time allowed : Two hours
Attempt all FOUR questions
1. Let $X$ be a set, and let $\leq$ be a total order on $X$. We say that $X$ is dense if for any two $x, y \in X$ with $x < y$ there exists $z \in X$ such that $x < z < y$. In what follows $\leq$ always denotes the usual ordering on rational numbers.

(i) For each of the following sets determine whether it is dense or not: (a) $\mathbb{Z}$ (integers); (b) $\mathbb{Q}$ (rationals); (c) $\{ x \in \mathbb{Q} : 0 \leq x \leq 1 \}$; (d) $\{ x \in \mathbb{Q} : 0 \leq x \leq 1 \} \cup \{ x \in \mathbb{Q} : 2 \leq x \leq 3 \}$. Justify your assertions. [4]

(ii) Let $a$ and $b$ be rational numbers satisfying $a < b$. To which of the following three sets does the number $\frac{2a + 3b}{5}$ belong: $A = \{ x \in \mathbb{Q} : x < a \}$, $B = \{ x \in \mathbb{Q} : a < x < b \}$ or $C = \{ x \in \mathbb{Q} : b < x \}$? Justify your answer. [1]

(iii) Consider the set

$$A = \{ \frac{a}{5^n} : a, n \in \mathbb{Z}, n \geq 0 \}.$$  
Is $A$ dense? Justify your answer. [2]

(iv) Prove that $r = 1$ is the only positive rational number such that $r + \frac{1}{r}$ is an integer. [2]

(v) How many positive real numbers $x$ are there such that $x + \frac{1}{x}$ is an integer? Your answer should be one of: 0, 1, 2, 3, ..., countably infinite, or uncountably infinite, and you should justify it. [3]

2. A Dedekind cut is a set $\emptyset \neq A \subseteq \mathbb{Q}$ satisfying the following three conditions:

(C1) $A$ is bounded above.

(C2) $A$ has no maximum.

(C3) $A$ is closed downwards, i.e. if $x \in A$ and $y \leq x$ then $y \in A$.

(i) For each of the following sets determine whether or not it is a Dedekind cut:

(a) $O = \{ x \in \mathbb{Q} : x < 0 \}$;

(b) $B = \{ x \in \mathbb{Q} : x^2 < 2 \}$;

(c) $C = \{ x^2 : x \in \mathbb{Q}, x < 2 \}$;

(d) $D = \mathbb{Z}$.

You do not need to provide rigorous proofs of your assertions, but for each set that is not a Dedekind cut state at least one condition (C1), (C2), (C3) which it fails. [4]

(ii) Find a set $E \subseteq \mathbb{Q}$ which satisfies (C1) and (C3) but fails (C2). [1]
The addition of Dedekind cuts is defined as follows:

\[ A + B = \{a + b : a \in A, \ b \in B\}. \]

In what follows you may assume without proof that if \( A \) and \( B \) are cuts then so is \( A + B \).

(iii) Prove the commutative law for addition: \( A + B = B + A \). \[1\]

(iv) Let \( A \) be a cut, and let \( t \) be an arbitrary positive rational number. Prove that there exists \( a \in A \) such that \( a + t \notin A \). Moreover, prove that \( a \) can be chosen so that \( a + t \) is not the least upper bound for \( A \). \[3\]

(v) Explain what is wrong with defining the negative cut corresponding to the cut \( A \) by

\[ -A = \{-a : a \in A\}. \]

The negative of a cut \( A \) is actually defined as:

\[ -A = \{-m : m \text{ is an upper bound, but not least, for } A\}. \]

In what follows you may use without proof the fact that if \( A \) is a cut then so is \(-A\).

(vi) Prove that \( A + (-A) = O = \{x \in \mathbb{Q} : x < 0\} \) for every cut \( A \). You may wish to proceed as follows. To prove the direct inclusion, take \( a \in A \) and \( b \in -A \) and show that \( a + b < 0 \). For the converse inclusion first let \( c < 0 \) be arbitrary, then let \( t = -c \), and then use (iv). \[3\]

3. (i) Express the number with the periodic decimal expansion 0.321212121\ldots in the form \( \frac{m}{n} \), \( m, n \in \mathbb{Z} \). \[2\]

(ii) Let \( r \) be the number 0.a_1a_2a_3\ldots, where \( a_n \) is the last digit of \( n \) for every \( n \in \mathbb{N} \). Is \( r \) rational or irrational? Justify your answer. \[1\]

(iii) Let \( s \) be the number 0.a_1a_2a_3\ldots, where \( a_n \) is the first digit of \( n \) for every \( n \in \mathbb{N} \). Is \( s \) rational or irrational? Justify your answer. \[2\]

Let \( A \subseteq \mathbb{R} \) be the set of all real numbers with (infinite) decimal expansion of the form 0.d_1d_2d_3\ldots where \( d_i \in \{2, 3\} \) for all \( i \in \mathbb{N} \).

(iv) Prove that \( A \) contains infinitely many rational numbers, and also infinitely many irrational numbers. \[2\]
(v) Does $A$ have a minimum (smallest element)? If so, what is it? Does $A$ have 
a maximum (largest element)? [2]

(vi) Is $A$ dense? Justify your answer. (For a reminder of the definition of density 
see Question 1.) [2]

(vii) Give an example of a set $B$ of real numbers such that $B \subseteq [0, 1]$, $B$ is dense, 
countable and consists entirely of irrational numbers. Justify your assertions. [2]

4. (i) Prove that the open interval $(0, 1)$ is uncountable. [3]

(ii) State (without proof) the Schr"oder–Bernstein Theorem. [1]

(iii) Using the Schr"oder–Bernstein Theorem, or otherwise, prove that the open 
interval $(0, 1)$ and the open square $(0, 1) \times (0, 1)$ have the same cardinality. [3]

(iv) Determine which of the following three assertions is true:

(a) $\mathbb{C}$ (the set of all complex numbers) has cardinality less that $\mathbb{R}$ (the set of 
all real numbers);

(b) $\mathbb{R}$ has cardinality less than $\mathbb{C}$;

(c) $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality.

Justify your answer. You may assume without proof the existence of a bijection 
between the set of real numbers $\mathbb{R}$ and the open interval $(0, 1)$. [2]

(v) The factorization of a natural number into a product of primes is unique up 
to reordering. That is, for all $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k \in \mathbb{N} \cup \{0\}$:

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k,$$

where $p_1, \ldots, p_k$ are distinct prime numbers.

Using this fact and the Schr"oder–Bernstein Theorem, or otherwise, prove that the set 

$$\underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_n$$

is countable for all $n \in \mathbb{N}$. [3]